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A PROBABILISTIC APPROACH TO EXPRESSIONS OF STIRLING NUMBERS OF THE FIRST KIND

Hajime YAMATO* and Tsunehiro FUJISAKI**

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Abstract

Using a sequence of independent random variables which take on one of the values 0 and 1 with prescribed probabilities, we give probabilistic proofs of two expressions of Stirling numbers of the first kind.

1. Introduction and Summary

Some theorems that can be stated without reference to probability nonetheless have simple probabilistic proofs. Bernstein's approach to the Weierstrass approximation theorem is based on the binomial distribution. (See Billingsley [1], p. 72.) Some combinatorial identities have also probabilistic proofs. This work gives probabilistic proofs of two expressions of Stirling numbers of the first kind.

Stirling numbers of the first kind are the numbers $s(n, k)$ such that

$$(x)_n = \sum_{k=0}^{n} s(n, k)x^k \qquad \text{for} \quad n = 1, 2, \ldots,$$

where $(x)_n = x(x-1)\cdots(x-n+1)$. By convention, we take $s(n, 0) = 0$ for $n > 0$, $s(0, 0) = 1$, and $s(n, k) = 0$ for $k > n$. Its absolute value $|s(n, K)|$ is equal to $(-1)^{n+k}s(n, k)$.

Stirling numbers of the first kind $s(n, k)$ has the following well-known expressions:

$$|s(n, k)| = \sum\nolimits^{+}_{1 \leqq i_1 < \cdots < i_{n-k} \leqq n-1} i_1 \cdots i_{n-k},$$

where the summation Σ^+ takes place over all positive integers $i_1, \ldots, i_{n-k}$ satisfying $1 \leqq i_1 < \cdots < i_{n-k} \leqq n-1$.

$$|s(n, k)| = \frac{n!}{k!} \sum\nolimits^{+}_{r_1 + \cdots + r_k = n} \frac{1}{r_1 r_2 \cdots r_k}, \tag{1.1}$$

* Department of Mathematics, Faculty of Science, Kagoshima University, Kagoshima 890, Japan
** Kagoshima National College of Technology, Kagoshima 899–51, Japan

where the summation Σ^+ takes places over all positive integers $r_1, \ldots, r_k$ satisfying $r_1 + \cdots + r_k = n$.

$$|s(n, k)| = \sum_{\Sigma k_j = k,\ \Sigma j k_j = n} \frac{n!}{\prod_j j^{k_j} \cdot k_j!}, \tag{1.2}$$

where the summation takes over all nonnegative integers $k_1, \ldots, k_n$ satisfying $k_1 + \cdots + k_n = k$ and $k_1 + 2k_2 + \cdots + nk_n = n$. (See for example Charalambides and Singh [2] and Comtet [3].)

Using a sequence of independent random variables which take on one of the values zero and one with prescribed probabilities, Yamato [8] presents a probabilistic proof of the first expression. By making use of Dirichlet process Yamato [7] gives a probabilistic proof of the expression (1.2).

Our aim is to yield probabilistic proofs of expressions of Stirling numbers of the first kind given by (1.1) and (1.2) using the sequence of independent and discrete random variables.

2. Proofs of expressions of Stirling numbers

Let $B_1, B_2, \ldots$ be a sequence of discrete random variables as follows: $B_1, B_2, \ldots$ are independent and for $M > 0$ and $j = 1, 2, \ldots$ B_j take on one of the values 1 and 0 with probabilities $(j-1)/(M+j-1)$, and $M/(M+j-1)$, respectively. That is

$$P(B_j = 0) = (j-1)/(M+j-1), \quad P(B_j = 1) = M/(M+j-1) \quad \text{for} \quad j = 1, 2, \ldots \tag{2.1}$$

The above sequences of random variables are for example given by an urn model (Yamato [8]) and the sequence of new observations from a distribution having Dirichlet process (Korwar and Hollander [4]).

Let us put $B(n) = B_1 + \cdots + B_n$ for $n = 1, 2, \ldots,$ which is a random variable taking on one of the values $1, \ldots, n$. $B(1)$ is equal to one with probability one. Since $B_1, B_2, \ldots$ are independent and take on one of the values 0 and 1, $B(n)$ satisfies the following recursive relation.

Lemma 2.1. For $k = 1, \ldots, n$,

$$P(B(n+1) = k) = P(B(n) = k)P(B_{n+1} = 0) + P(B(n) = k-1)P(B_{n+1} = 1), \tag{2.2}$$

$$P(B(n+1) = n+1) = P(B(n) = n)P(B_{n+1} = 1). \tag{2.3}$$

The random variable $B(n)$ has the probability distribution related to Stirling numbers of the first kind as follows (Sibuya [5] and Yamato [8]).

Lemma 2.2.

$$P(B(n)=k)=|s(n,k)|M^k/\langle M\rangle_n \quad \text{for} \quad k=1,\dots,n \quad \text{and} \quad n=1,2,\dots, \tag{2.4}$$

where $\langle M\rangle_n = M(M+1)\cdots(M+n-1)$.

Proof of expression (1.1). We prove this by induction using the probabilistic relations (2.2) and (2.3). Since $P(B(1)=1)=1$, by Lemma 2.2 we have, $|s(1,1)| = 1$. On the other hand the right-hand side of (1.1) with $n=k=1$ is equal to 1. Thus the expression (1.1) holds for $n=1$.

Now we assume the expression (1.1) holds for $k=1,\dots,n$ with $n \geqq 1$. We shall show that (1.1) holds for $k=1,\dots,n+1$ with $n+1$.

For $k=1,\dots,n$, by (2.2) and (2.4) we have

$$P(B(n+1)=k) \tag{2.5}$$

$$=\frac{n!}{k!}\left\{n \sum_{r_1+\cdots+r_k=n}^{+} \frac{1}{r_1\cdots r_k}+k \sum_{r_1+\cdots+r_{k-1}=n}^{+} \frac{1}{r_1\cdots r_{k-1}}\right\}\frac{M^k}{\langle M\rangle_{n+1}},$$

where the summation Σ^+'s takes places over all positive integers $r_1,\dots,r_k$ and $r_1,\dots,r_{k-1}$ satisfying $r_1+\cdots+r_k=n$ and $r_1+\cdots+r_{k-1}=n$, respectively.

On the other hand, for $n=1,2,\dots$ and $k=1,\dots,n$ we have

$$(n+1)\sum_{r_1+\cdots+r_k=n+1}^{+}\frac{1}{r_1\cdots r_k}=\sum_{j=1}^{k}\sum_{r_1+\cdots+r_k=n+1}^{+}\frac{r_j}{r_1\cdots r_k}. \tag{2.6}$$

Separating the cases of $r_j=1$ and $\geqq 2$ for $j=1,\dots,k$,

$$\text{R.H.S. of (2.6)}=\sum_{j=1}^{k}\left\{\sum_{r_1^*+\cdots+r_k^*=n}^{+}\frac{r_j^*}{r_1^*\cdots r_k^*}\right.$$

$$\left.+\sum_{r_1+\cdots r_{j-1}+r_{j+1}+\cdots+r_k=n}^{+}\frac{1}{r_1\cdots r_{j-1}r_{j+1}\cdots r_k}\right\}$$

$$=n\sum_{r_1+\cdots+r_k=n}^{+}\frac{1}{r_1\cdots r_k}+k\sum_{r_1+\cdots+r_{k-1}=n}^{+}\frac{1}{r_1\cdots r_{k-1}}.$$

By applying the above relation to the right-hand side of (2.5),

$$P(B(n+1)=k)=\frac{(n+1)!}{k!}\sum_{r_1+\cdots+r_k=n+1}^{+}\frac{1}{r_1\cdots r_k}\cdot\frac{M^k}{\langle M\rangle_{n+1}}.$$

Thus by Lemma 2.2, for $k=1,\dots,n$ we

$$|s(n+1,k)|=\frac{(n+1)!}{k!}\sum_{r_1+\cdots+r_k=n+1}^{+}\frac{1}{r_1\cdots r_k}.$$

Since $|s(n, n)| = 1$, by (2.3) and (2.4) we have for $k = n + 1$

$$P(B(n+1) = n+1) = \frac{M^n}{\langle M \rangle_n} \cdot \frac{M}{M+n} = \frac{M^{n+1}}{\langle M \rangle_{n+1}}.$$

Therefore by Lemma 2.2, we have $|s(n+1, n+1)| = 1$. Thus the expression (1.1) holds for $k = 1, \dots, n+1$ with $n+1 > 0$ and is proved by the induction.

Before proving the expression (1.2) we quote the following lemma, which is shown in the proof of Corollary of Theorem 4 of Sibuya, Kawai and Shida [6].

Lemma 2.3. For $n = 1, 2, \dots$ and $k = 1, \dots, n$,

$$\sum_{\Sigma k_j = k,\ \Sigma j k_j = n+1} f(k_1, k_2, \dots; n+1) \tag{2.7}$$

$$= \sum_{\Sigma k_j = k-1,\ \Sigma j k_j = n} f(k_1, k_2, \dots; n) + n \sum_{\Sigma k_j = k,\ \Sigma j k_j = n} f(k_1, k_2, \dots; n)$$

where $f(k_1, k_2, \dots; n) = n!/[\prod_j j^{k_j} \cdot k_j!]$ for $k_1, k_2, \dots \geqq 0$ and $n = 1, 2, \dots$ and the summation Σ's take over all nonnegative integers $k_1, k_2, \dots$ satisfying $k_1 + k_2 + k_3 + \cdots = k$ and $k_1 + 2k_2 + 3k_3 + \cdots = n+1$ and so on.

Proof of expression (1.2). We prove this by induction using the probabilistic relations (2.2) and (2.3). As stated at the beginning of the proof of (1.1), $|s(1, 1)| = 1$. On the other hand the right-hand side of (1.2) with $n = k = 1$ is equal to 1. Thus the expression (1.2) holds for $n = 1$.

Now we assume the expression (1.2) holds for $k = 1, \dots, n$ with $n \geqq 1$. We shall show that (1.2) holds for $k = 1, \dots, n+1$ with $n+1$.

For $k = 1, \dots, n$, by (2.2) and (2.4) we have

$$P(B(n+1) = k)$$

$$= \left\{ \sum_{\Sigma k_j = k-1,\ \Sigma j k_j = n} f(k_1, k_2, \dots; n) + n \sum_{\Sigma k_j = k,\ \Sigma j k_j = n} f(k_1, k_2, \dots; n) \right\} M^k / \langle M \rangle_{n+1}.$$

By applying Lemma 2.3 to the right-hand side of the above,

$$P(B(n+1) = k) = \sum_{\Sigma k_j = k,\ \Sigma j k_j = n+1} f(k_1, k_2, \dots; n+1) \frac{M^k}{\langle M \rangle_{n+1}}.$$

Thus by Lemma 2.2, for $k = 1, \dots, n$ we

$$|s(n+1,k)| = \sum_{\Sigma k_j = k,\ \Sigma j k_j = n+1} \frac{(n+1)!}{\prod_j j^{k_j} \cdot k_j!}.$$

Since $|s(n,n)| = 1$ with $|s(n,k)|$ given by (1.2), using (2.3) and (2.4) we have $|s(n+1, n+1)| = 1$ by the similar discussion to the last of the proof of (1.1). Thus the expression (1.2) holds for $k = 1, \dots, n+1$ with $n+1 \geqq 2$ and is proved by the induction.

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References

[1] P. Billingsley, Probability and Measure, Wiley, New York, 1979.
[2] Ch. A. Charalambides and J. Singh, A review of the Stirling numbers, their generalization and statistical applications, Commun. Statist, -Theory Meth., **17** (1988), 2533–2593.
[3] L. Comtet, Advanced Combinatorics, D. Reidel, Dordrecht, 1974.
[4] R. M. Korwar and M. Hollander, Contributions to the theory of Dirichlet processes, Ann. Probab., **1** (1973), 705–711.
[5] M. Sibuya, Stirling family of probability distributions, A survey, Japan. J. Appl. Statist., **15** (1986), 131–146 (in Japanese).
[6] M. Sibuya, T. Kawai and K. Shida, Equipartition of particles forming clusters by inelastic collisions, Physica A, **167** (1990), 676–689.
[7] H. Yamato, Probabilistic proofs of relations with Stirling numbers of the first kind by Dirichlet process, Statist. & Probab. Letters, 10 (1990), 189–193.
[8] H. Yamato, A probabilistic approach to Stirling numbers of the first kind, Commun. Statist. -Theory Meth., **19** (1990), 3915–3923.