On the Series of Independent Functions

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0. Introduction

The present paper deals with the series of independent real valued measurable functions on a probability space (X, S, μ). Most of the series of independent functions are discussed in [1].

1. Basic Notation and Terminology

Throughout this work the triple (X, S, μ) will denote a probability space. If E is a finite or infinite set of real Valued measurable functions on a probability space (X, S, μ), the functions of the set E are (stochastically) independent if

$$\mu(\bigcap_{i=1}^{n}\{x: f_i(x)\in M_i\}) = \prod_{i=1}^{n}\mu(\{x: f_i(x)\in M_i\})$$

for every finite subset $\{f_i: i = 1,\ldots\ldots, n\}$ of distinct functions in E and every finite class $\{M_i: i = 1,\ldots\ldots,n\}$ of Borel sets on the real line. [1]

2. Series of Independent Functions

Before proving the main result we will state following several lemmas.

Lemma 1.

If $\{E_n\}$ is a sequence of measurable sets, and $\sum_{n=1}^{\infty}\mu(E_n)<\infty$, then $\mu(\limsup_n E_n) = 0$.

proof.

Since $\{\bigcup_{j=n}^{\infty}E_j\}$ is a decreasing sequence for $n=1, 2,\ldots\ldots$,

$$\mu(\limsup_n E_n) = \mu(\lim_n \bigcup_{j=n}^{\infty} E_j) = \lim_n\mu(\bigcup_{j=n}^{\infty} E_j)\leqq\lim_n\sum_{j=n}^{\infty}\mu(E_j) = 0.$$

Lemma 2.

If $\{f_n\}$ is a sequence of independent functions and C is a positive constant such that $|f_n(x)|\leqq C$ *a. e.*, $n=1, 2,\ldots\ldots$, then $\sum_{n=1}^{\infty}f_n(x)$ converges *a. e.* if and only if both the series $\sum_{n=1}^{\infty}f_n d\mu$ and $\sum_{n=1}^{\infty}\sigma^2(f_n)$ are convergent. (See [1] Sec. 46, Theorem D).

Lemma 3. (Borel-Cantelli lemma)

If $\{E_n\}$ is a sequence of independent sets, then $\mu(\limsup_n E_n) = 0$, if and only if $\sum_{n=1}^{\infty}\mu(E_n)<\infty$.

Proof.

If $\sum_{n=1}^{\infty}\mu(E_n)<\infty$, it follows from lemma 1 that $\mu(\limsup_n E_n) = 0$. To prove the converse, let χ_n be the characteristic function of E_n. It is clear that $\mu(\limsup_n E_n) = 0$ is equivalent to $\mu(\liminf_n E_n')= 1$, (*i. e.* that $E_n'\ni x$ *a. e.* for $n\geqq n_0$). Since $\chi_n(x)=0$ *a. e.* for $n\geqq n_0$, $\sum_{n=1}^{\infty}\chi_n(x)$ converges *a. e.*. Therefore it follows from lemma 2 that $\sum_{n=1}^{\infty}\int\chi_n d\mu = \sum_{n=1}^{\infty}\mu(E_n)<\infty$ and the converse is also true.

Lemma 4.

If $\{f_n\}$ and $\{g_n\}$ are sequences of independent functions and we put $\{x: f_n(x) \neq g_u(x)\} = E_n$, $n=1, 2, \ldots\ldots$, if $\sum_{n=1}^{\infty} \mu(E_n) < \infty$, then a necessary and sufficient condition for the convergence *a. e.* of the series $\sum_{n=1}^{\infty} f_n(x)$ is the existence of an equivalent sequence $\{g_n\}$ of independent functions with finite variance such that the series $\sum_{n=1}^{\infty} \int g_n d\mu$ and $\sum_{n=1}^{\infty} \sigma^2(g_n)$ are convergent.

Proof.

Since $\sum_{n=1}^{\infty} \mu(E_n) < \infty$, two sequences $\{f_n\}$ and $\{g_n\}$ are equivalent in the sense of Khintchine. By the assumption, it follows from lemma 1 that $\mu(\limsup_n E_n) = 0$. And hence $\mu(\limsup E_n') = 1$ *i. e.* $f_n(x) = g_n(x)$ *a. e.* for $n \geqq n_0$. Therefore if $\sum_{n=1}^{\infty} f_n(x)$ converges *a. e.*, then $\sum_{n=1}^{\infty} g_n(x)$ converges *a. e.*. The desired result follows from lemma 2. (See [1] Sec. 46, (5)).

THEOREM.

If $\{f_u\}$ and $\{g_n\}$ are sequences of independent functions and we put $\{x: f_n(x) \neq g_n(x)\} = E_n$, $n = 1, 2, \ldots\ldots$, and if $\mu(\limsup_n E_n) = 0$, then a necessary and sufficient condition for the convergence a. e. of the series $\sum_{n=1}^{\infty} f_n(x)$ is that the series $\sum_{n=1}^{\infty} \int g_n d\mu$ and $\sum_{n=1}^{\infty} \sigma^2(g_n)$ are convergent.

Proof.

If $\{h_n\}$ is a sequence of functions defined by $h_n(x) = f_n(x) - g_n(x)$ $n = 1, 2, \ldots\ldots$, then the sequence $\{h_n\}$ is independent. Therefore $\{E_n\}$ is a sequence of independent sets. Since $\mu(\limsup E_n) = 0$, it follows from lemma 3 that $\sum_{n=1}^{\infty} \mu(E_n) < \infty$, and thus the theorem's result follows from lemma 4.

Reference

1. P. R. Halmos Measure Theory Van Nostrand, 1950.