A generalization of Lie groups and symmetric spaces

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Abstract

In 1954, an important notion of reductive homogeneous spaces was introduced by K. Nomizu in [12]. For a differentiable manifold with an affine connection, we shall denote the torsion tensor field and the curvature tensor field by T and R, and the covariant differentiation by ∇. As is well known, Lie groups have the so-called $(-)$-connection (see É. Cartan and J. A. Schouten [2]), with the properties that $\nabla T=0$, and $R=0$. Similarly, affine symmetric spaces are characterized locally by an affine connection on them with the properties that $T=0$, and $\nabla R=0$ (see, for instance, [11], Theorem 4.9, p. 114). So, to consider such manifolds with an affine connection that $\nabla T=0$, and $\nabla R=0$, is natural. These manifolds are, precisely, locally reductive spaces in K. Nomizu [12].

Since the discovery of symmetric spaces by É. Cartan in [1], many investigations have been done. Among them, the work (Dissertation) of 0. Loos in 1966 (see [11]) is epock-making, in the historical view of geometry; axiomatic, differential geometric, and algebraic. 0. Loos has succeeded to characterize symmetric spaces by a multiplication on them. Following 0. Loos's idea, locally reductive spaces and related spaces have been investigated by M. Kikkawa ([6], [7], [9], [10], and the other papers), intensively, from the loop theoretic point of view. Our study owes very much to his investigation. A slight generalization of the notion of differentiable homogeneous systems due to M. Kikkawa [7], is tried (Definition 1), and can be shown that $\nabla T=0$, and $\nabla R=0$ for the canonical connection on these spaces (Theorem 1). The key lemma (Lemma 3) in the proof of this result is also found by M. Kikkawa in [10] independently.

The tangent algebra of locally reductive spaces (Lie triple algebra or general Lie triple system) was constructed by K. Yamaguti in [13]. An important theorem that any Lie triple algebra can be imbedded canonically into the corresponding Lie algebra has been shown essentially in K. Nomizu [12], pp. 61–62 (see, also, K. Yamaguti [13], Proposition 2.1, p. 158, and M. Kikkawa [5], Proposition 1, p. 2). This algebra has been studied by K. Yamaguti [14], and others, especially, recent Lie algebraic approach by M. Kikkawa (for instance [8]) seems to show the mathematical reality of this algebra. Our final aim is to find such a (triple) multiplication on manifolds that a simply connected manifold with this multiplication corresponds, bijectively, to a Lie triple algebra up to isomorphism (cf. [4]).

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§1. Parallelizable spaces

In this paper, we will investigate the following spaces with a triple multiplication.

Definition 1. Let M be a finite dimensional differentiable manifold of class C^{∞}, and let $\eta : M\times M\times M\to M$ be a differentiable triple multiplication of class C^{∞}. If η satisfies the following conditions (P_1), (P_2), (P_3), and (P_4), then M is called a *parallelizable space* with an extensive constant k (abbreviated as p-space or p-space with k).

(P_1) $\eta(x, x, y)=y$,

(P_2) $\eta(x, y, \eta(y, x, z))=z$,

(P_3) $\eta(x, y, \eta(u, v, w))=\eta(\eta(x, y, u), \eta(x, y, v), \eta(x, y, w))$,

(P_4) for each $x\in M$, in local coordinates,

$$\frac{\partial \eta_i}{\partial y_j}(x, y, z)_{y=x,\ z=x}=k\delta^i_j \qquad \text{where } k \text{ is a non-zero constant.}$$

Remark 1. The condition (P_4) is independent of the choice of local coordinates.

Remark 2. This definition is a slight extension of that of differentiable homogeneous systems introduced by M. Kikkawa in [7] (see Example 1 below).

First of all, we shall note some notational conventions.

Notational conventions

i) In general we follow the notation and terminology of 0. Loos [11].

All manifolds are finite dimensional differentiable manifolds of class C^{∞}, and all mappings between these manifolds are C^{∞}-class.

For a manifold M, $F(M)$ denotes the set of all real valued C^{∞}-functions on M, $T_e(M)$ denotes the tangent space to M at $e\in M$, and $T^{\infty}_e(M)$ denotes the space of all higher order tangent vectors at e (see p. 5).

ii) We use Einstein's summation convention.

iii) A parallelizable space M with triple multiplication η and extensive constant k is denoted by (M, η, k) or (M, η).

iv) For the triple multiplication $\eta=\eta(x, y, z)$ on a p-space (M, η, k), x is called the first variable, y the second, and z the third. As notation of partial derivatives of η, we always use x for the first variable, for instance

$$\frac{\partial \eta_i}{\partial x_j}(a, b, c), \quad \frac{\partial \eta_p}{\partial x_q}(u, v, w), \ \ldots,$$

y for the second variable, and z for the third variable.

In these calculations, also, we always use the same local coordinatate system for each M, when it is possible.

v) We use freely calculations with local coordinates, when these meanings are clear from contexts.

The following is basic properties of p-spaces.

PROPOSITION 1. *For a p-space* (M, η, k), *we have*

$$\frac{\partial \eta_i}{\partial x_j}(x, x, x) = -k\delta^i_j \tag{1.1}$$

$$\frac{\partial \eta_i}{\partial y_j}(x, x, x) = k\delta^i_j \tag{1.2}$$

$$\frac{\partial \eta_i}{\partial z_j}(x, x, y) = \delta^i_j \tag{1.3}$$

$$\frac{\partial^2 \eta_i}{\partial x_j \partial z_k}(x, x, y) = -\frac{\partial^2 \eta_i}{\partial y_j \partial z_k}(x, x, y) \tag{1.4}$$

$$\frac{\partial^2 \eta_i}{\partial z_j \partial z_k}(x, x, y) = 0 \tag{1.5}$$

Definition 2. Let (M, η, k) and (N, η, k) be two p-spaces with the same extensive constant k. A mapping $\phi: M \to N$ is called a *homomorphism* of M into N, if

(H) $\phi(\eta(x, y, z)) = \eta(\phi(x), \phi(y), \phi(z))$

is satisfied for all $x, y, z \in M$.

A homomorphism ϕ is called an *isomorphism* of M onto N, if ϕ is a diffeomorphism of M onto N. In the case $M=N$, an isomorphism of M onto M is called an *automorphism* of M.

Remark 3. If $\partial\phi_i/\partial x_j(x) \neq 0$ for some i, j and for some $x \in M$, then operating $Y = \partial/\partial y_j$ for (H), we have

$$\frac{\partial \phi_i}{\partial x_a}(\eta(x, y, z))\frac{\partial \eta_a}{\partial y_j}(x, y, z) = \frac{\partial \eta_i}{\partial y_a}(\phi(x), \phi(y), \phi(z))\frac{\partial \phi_a}{\partial y_j}(y),$$

therefore, putting $y=z=x$, we see that the extensive constants of M and N are equal to each other (withhout this assumption in the above defintion).

The following examples are a motivation for our definition of p-spaces.

Example 1. Differentiable homogeneous systems.

These spaces are the spaces with a triple multiplication η, which satisfies the conditions (P_1), (P_2), (P_3), and

(H_4) $\eta(x, y, x) = y$.

M. Kikkawa has studied these spaces and related spaces intensivly ([6], [7], [9], [10], and the other papers).

Example 2. Lie groups.

Let M be a Lie group. If the triple multiplication η is defined on M by

$$\eta(x, y, z) = y \cdot x^{-1} \cdot z, \tag{1.6}$$

then, M becomes a homogeneous system, therefore, a p-space. This p-space will be said to be the parallelizable space of the Lie group M. This example is a typical example of homogeneous systems.

Example 3. Symmetric spaces.

A symmetric space is a manifold M with a multiplication $\mu: M \times M \to M$, written as $\mu(x, y) = x \bigcirc y$, which satisfies the following conditions (see [11], p. 63):

(S_1) $x\bigcirc x=x$, (S_2) $x\bigcirc(x\bigcirc y)=y$,

(S_3) $x\bigcirc(y\bigcirc z)=(x\bigcirc y)\bigcirc(x\bigcirc z)$,

(S_4) every x has a neighborhood U such that $x\bigcirc y=y$, $y\in U$ imply $y=x$.

Let M be a symmetric space. If the triple multiplication η on M is defined by

$$\eta(x, y, z)=x\bigcirc(y\bigcirc z), \tag{1.7}$$

then, M becomes a p-space with -2, which will be said to be the parallelizable space of the symmetric space M.

PROOF. The conditions (P_1), (P_2), and (P_3) are easily checked. From Lemma 2 below, we see that the mapping $y\to x\bigcirc y$ has the Jacobian matrix $(-\delta^i_j)$ at x, and the mapping $y\to y\bigcirc x$ has the Jacobian matrix $(2\delta^i_j)$ at x. Therefore, the mapping $y\to\eta(x, y, x)=x\bigcirc(y\bigcirc x)$ has the Jacobian matrix $(-2\delta^i_j)$ at x.

Remark 4. In a normal coordinate system at x, we have $\eta_i(x, y, x)=-2y_i$, our assertion follows, also, from this.

Definition 3. Let (M, η) be a p-space.

i) The mapping $\eta(x, y)$: $M\to M$ defined by $\eta(x, y)z=\eta(x, y, z)$, is called *displacement* (of direction from x to y).

ii) The mapping H_x: $M\to M$ defined by $H_xy=\eta(x, y, x)$, is called *homothety* (with center x).

iii) The mapping S_x: $M\to M$ defined by $S_xy=\eta(x, y, y)$, is called *similarity* (with center x).

Conditions (P_2) and (P_3) mean that any displacement $\eta(x, y)$ is an automorphism of M. And, if M is the p-space of a symmetric space S, the similarity S_x is the symmetry around x on S ([11], p. 64).

The following interesting characterization of Lie groups dues to M. Kikkawa ([7], Prop. 2, p. 20):

PROPOSITION 2. *Let (M, η) be a p-space. M is that of a Lie group, if and only if the following conditions are satisfied.*

i) $H_x=id$ *(identity mapping),* for all $x\in M$

ii) $\eta(y, z)\eta(x, y)=\eta(x, z)$, for all $x,y,z\in M$.

PROPOSITION 3. *Let (M, η) be a p-space. M is that of a symmetric space, if and only if the following conditions are satisfied.*

i) $S^2_x=id$ *(identity mapping),* for all $x\in M$

ii) $\eta(x, y)\eta(y, z)=\eta(x, z)$, for all $x,y,z\in M$.

LEMMA 1. *For a p-space (M, η), the conditions* i) *and* ii) *in the above Proposition* 3 *are equivalent to*

iii) $\eta(x, y)=S_xS_y$, for all $x,y\in M$.

PROOF. From ii), $\eta(x, y)\eta(y, z)z=\eta(x, z)z$, i.e. $\eta(x, y)S_y=S_x$, so $\eta(x, y)S^2_y=S_xS_y$. Therefore, from i) $\eta(x, y)=S_xS_y$.

Conversely, from iii) $id=\eta(x, x)=S_xS_x$, and $\eta(x, y)\eta(y, z)=S_xS_yS_yS_z=S_xS_z=\eta(x, z)$.

PROOF OF PROPOSITION 3. It is easily seen that a symmetric space satisfies the condition iii), from (1.7).

Now, we assume that a p-space (M, η) satisfies the conditions i), ii), and iii) above. Then M becomes a symmetric space with respect to the natural multiplication $x\bigcirc y=S_xy$.

The conditions (S_1) and (S_2) are obviously satisfied.

As to the condition (S_3), we have

$$x \bigcirc (y \bigcirc z) = S_x S_y z = \eta(x, y) z,$$

on the other hand, we have

$$\begin{aligned}
&(x \bigcirc y) \bigcirc (x \bigcirc z) \\
&= \eta(\eta(x, y, y), \eta(x, z, z), \eta(x, z, z)) \\
&= \eta(x, y)\eta(y, \eta(y, x)\eta(x, z, z), \eta(y, x)\eta(x, z, z)) \\
&= \eta(x, y)\eta(y, \eta(y, z, z), \eta(y, z, z)) \\
&= \eta(x, y) S_y S_y z \\
&= \eta(x, y) z,
\end{aligned}$$

therefore, they are equal to each other.

Finally, the following calculation shows that the local condition (S_4) is satisfied. From iii),we have

$$\eta(x, y, z) = \eta(x, \eta(y, z, z), \eta(y, z, z)),$$

therefore

$$\begin{aligned}
\frac{\partial \eta_i}{\partial y_j}(x, y, z) = {} & \frac{\partial \eta_i}{\partial y_a}(x, \eta(y, z, z), \eta(y, z, z)) \frac{\partial \eta_a}{\partial x_j}(y, z, z) \\
& + \frac{\partial \eta_i}{\partial z_a}(x, \eta(y, z, z), \eta(y, z, z)) \frac{\partial \eta_a}{\partial x_j}(y, z, z),
\end{aligned}$$

evaluating at $y = z = x$, and using (1.1), (1.2), and (1.3),

$$k \delta^i_j = -k^2 \delta^i_j - k \delta^i_j,$$

so, $k = -2$. Now, the mapping $y \to x \bigcirc y = \eta(x, y, y)$ has the elements of the Jacobian matrix at x,

$$\frac{\partial \eta_i}{\partial y_j}(x, x, x) + \frac{\partial \eta_i}{\partial z_j}(x, x, x) = -\delta^i_j.$$

This means that the mapping $y \to x \bigcirc y$ has, locally, no fixed points except $y = x$, by virtue of the mean value theorem.

§2. The canonical connection

Let M be an m-dimensional manifold. A tangent vector X of order k at $e \in M$ has, in local coordinates $(x_1, \ldots, x_m)$, an expression of the following form (including the term for $r = 0$)

$$X = \sum_{r=0}^{k} \sum_{i_1 \leq \cdots \cdots \leq i_r} \lambda\, i_1 \cdots\cdots i_r \, (\partial^r / \partial x_{i_1} \cdots\cdots \partial x_{i_r})_e.$$

These are called higher order tangent vectors at e, including that of order 1 and 0. And, the space of all higher order tangent vectors at e is denoted by $T^\infty_e(M)$.

Now, let N be also an n-dimensional manifold, and let $\phi : M \to N$ be a mapping of M into N. For a tangent vector X of order k at $e \in M$, the corresponding higher order tangent vector $T^k_e \phi(X)$ at $\phi(e) \in$

N is induced by

$$T_e^k\phi(X)f=X(f\circ\phi) \qquad for \quad f\in F(N). \tag{2.1}$$

In the sequel, we will simply write $T\phi$ or ϕ instead of $T_e^k\phi$, when there is no danger of ambiguity.

If ϕ is expressed, in local coordinates, by

$$y_i=\phi_i(x_1,\ldots,x_m) \qquad i=1,\ldots,n \tag{2.2}$$

then, for example, for $X=(\partial/\partial x_i)e$

$$\phi(X)f=(\frac{\partial}{\partial x_j}f(\phi(x)))e=\frac{\partial}{\partial y_a}f(\phi(e))\frac{\partial\phi_a}{\partial x_i}(e), \qquad \text{for} \quad f\in F(N)$$

so

$$\phi((\partial/\partial x_i)e)=\frac{\partial\phi_a}{\partial x_i}(e)(\partial/\partial y_a)_{\phi(e)}. \tag{2.3}$$

The above idea is very useful, and in the case where a multiplication $\mu(x,y)=x\cdot y$ is defined on M and $e\cdot e=e$ for some $e\in M$, is also worked. Namely, for any higher order tangent vectors X and Y at e, we can define their product $\mu(X,\ Y)=X\cdot Y$ by

$$(X\cdot Y)f=X\otimes Yf(x\cdot y), \qquad \text{for} \quad f\in F(M) \tag{2.4}$$

here, the tensor product $X\otimes Y$ means that X is operated for x, and Y for y (see [11], p. 48). This product $X\cdot Y$ is also a higher order tangent vector at e.

PROPOSITION 4. *The above product* (2.4) *has the following properties.*

i) *if* $e\cdot x=x$ *then* $e\cdot X=X$,

ii) *if* $x\cdot e=x$ *then* $X\cdot e=X$,

iii) *if* $(x\cdot y)\cdot z=x\cdot(y\cdot z)$ *then* $(X\cdot Y)\cdot Z=X\cdot(Y\cdot Z)$, *moreover, if* $f(x,y,z,\ldots)=g(x,y,z,\ldots)$ *then* $f(X,Y,Z\ldots)=g(X,Y,Z\ldots)$, *where f and g are some relations of* $x, y, z,\ldots,$

iv) *if left inverese* $x^{-1}=\phi(x)$ *of x with respect to e exists in the sense that* $\phi(x)\cdot x=e$ *and* $\phi(e)=e$, *and* $X\cdot e=kX$, $e\cdot X=rX$ *for* $X\in T_e(M)$ (k *aod* r *are constants,* $k\neq 0$)*,then* $\phi(X)=-r/k\ X$ *for* $X\in T_e(M)$.

In these statements i), ii), iii), *and* iv), *$x, y, z,\ldots$ denote any points in a neighborhood of e, and X, Y, $Z,\ldots$ denote any higher order tangent vectors at e, except X in* iv).

PROOF. i), ii), and iii) are obvious from definition.

As to iv), from $\mu(\phi(x),\ x)=e$ we have

$$\frac{\partial\mu_i}{\partial x_a}(x,\ e)_{x=\phi(e)}\frac{\partial\phi_a}{\partial x_j}(e)+\frac{\partial\mu_i}{\partial y_j}(\phi(e),y)_{y=e}=0,$$

i.e.

$$\frac{\partial\mu_i}{\partial x_a}(e,\ e)\frac{\partial\phi_a}{\partial x_j}(e)+\frac{\partial\mu_i}{\partial y_j}(e,\ e)=0.$$

So

$$k\frac{\partial \phi_i}{\partial x_j}(e)+r\delta^i_j=0,$$

which shows the property iv).

Example 4. For a Lie group M with unit element e, from i), ii), and iv) of the above Proposition 4, we see $\phi(X)=-X$, for $X\in T_e(M)$.

Example 5. Let M be a Lie group with unit element e, and let $\tilde{X}$, $\tilde{Y}$ be left invariant vector fields on M. The tangent vectors $X=\tilde{X}(e)$, and $Y=\tilde{Y}(e)$ at e are related to $\tilde{X}$ and $\tilde{Y}$ in the relations $\tilde{X}(x)=x\cdot X$, and $\tilde{Y}(x)=x\cdot Y$. The composition $[X, Y]$, in the Lie algebra of M, is defined by $[X, Y]=[\tilde{X}, \tilde{Y}](e)$, therefore for $f\in F(M)$,

$$[X, Y]f=(X\tilde{Y}-Y\tilde{X})f=X\otimes Y\ f(x\cdot y)-Y\otimes X\ f(y\cdot x),$$

i.e.

$$[X, Y]=X\cdot Y-Y\cdot X. \tag{2.5}$$

From the above Proposition 4 iii), the Jacobi identity of the Lie algebra of M is reduced to the associative law of M (see [11], p. 50).

For a triple multiplication on M, moreover an n-ple multiplication on M, similar products are defined. They have many useful properties. For instance, for a p-space (M,η,k) we have, corresponding to (1.1), (1.2), and (1.3) respectively

$$\eta(X, e, e)=-kX \tag{2.6}$$

$$\eta(e, X, e)=kX \tag{2.7}$$

$$\eta(e, e, X)=X \tag{2.8}$$

for all $X\in T_e(M)$.

Another property is the following.

PROPOSITION 5. *Let M be a manifold with an n-ple multiplication $\psi(x_1,\dots,x_n)$. If a triple multiplication $\phi(x, y, z)$ on M is introduced from ψ, by substituting x or y or z for each of these variables $x_1,\dots,x_n$, concretely (as $n=6$)*

$$\phi(x, y, z)=\psi(y, x, x, z, x, y),$$

and this ϕ satisfies $\phi(e, e, e)=e$ for some $e\in M$, then we have

$$\phi(X, Y, Z)=\psi(Y, X, e, Z, e, e)+\psi(Y, e, X, Z, e, e)+\psi(Y, e, e, Z, X, e)$$
$$+\psi(e, X, e, Z, e, Y)+\psi(e, e, X, Z, e, Y)+\psi(e, e, e, Z, X, Y), \tag{2.9}$$

for all X, $Y\in T_e(M)$ and $Z\in T^{\infty}_e(M)$.

PROOF. For $X=\partial/\partial x_i$, $Y=\partial/\partial x_j$, and $f\in F(M)$,

$$\phi(X, Y, Z)f = X\otimes Y\otimes Z\, f(\phi(x, y, z))$$

$$= Y\otimes Z\Big[\frac{\partial f}{\partial x_a}(\phi(e, y, z))\ \Big\{\frac{\partial \psi_a}{\partial x_i}(y, x, e, z, e, y)_{x=e} + \frac{\partial \psi_a}{\partial x_i}(y, e, x, z, e, y)_{x=e} + \frac{\partial \psi_a}{\partial x_i}(y, e, e, z, x, y)_{x=e}\Big\}\Big]$$

$$= \psi(Y, X, e, Z, e, Y)f + \psi(Y, e, X, Z, e, Y)f + \psi(Y, e, e, Z, X, Y)f.$$

And similarly we have

$$\psi(Y, X, e, Z, e, Y) = \psi(Y, X, e, Z, e, e) + \psi(e, X, e, Z, e, Y), \ldots.$$

Remark 5. For any m-ple multiplication ϕ on M, introduced from ψ similarly, we have the same property.

COROLLARY. *Let* (M, η) *be a p-space. Then*

$$\eta(X, e, Y) = -\eta(e, X, Y) \tag{2.10}$$

for all $X\in T_e(M)$ *and* $Y\in T_e^\infty(M)$.

PROOF. If ϕ is defined by $\phi(x, y) = \eta(x, x, y)$, then

$$\phi(X, Y) = \eta(X, e, Y) + \eta(e, X, Y).$$

On the other hand, $\phi(x, y) = y$ therefore $\phi(X, Y) = 0$.

Here, we shall mention some basic properties of differential geometry (see [11], p. 19).

In general, for an m-dimensional manifold M, an affine connection Γ on M and a covariant differentiation ∇ on M are related by the relation

$$\nabla_X Y = XY + \Gamma(X, Y) \tag{2.11}$$

where, X and Y are any vector fields on M.

In local coordinates $(x_1,\ldots,x_m)$, an affine connection Γ on M is expressed by

$$\Gamma(\partial/\partial x_i, \partial/\partial x_j) = -\partial^2/\partial x_i\partial x_j + \Gamma_{ij}^k(x)\partial/\partial x_k \tag{2.12}$$

here, Γ_{ij}^k are the Christoffel symbols.

From (2.11) and (2.12), we also have

$$\nabla_{\partial/\partial x_i}\partial/\partial x_j = \Gamma_{ij}^k(x)\,\partial/\partial x_k \tag{2.13}$$

Now, let (M, η, k) be a p-space. For any fixed point $e\in M$, we define the following multplication on M

$$e(x, y) = \eta(e, x, y) \qquad \text{for all } x, y\in M, \tag{2.14}$$

which has very important role in our investigation.

Definition 4. The multiplication (2.14) on a p-space (M, η, k) is called *the canonical multiplication* with base point e. This is usually denoted by $x\cdot y$, i.e.

$$x \cdot y = e(x, y) = \eta(e, x, y), \tag{2.14)'}$$

without the base point e, when there is no danger of confusion.

Remark 6. If a p-space M is that of a Lie group G, then the canonical multiplication $e(x, y)$ gives a Lie group structure on M, which is isomorphic to G by the left translation $L_e: G \to G$. Especially, if e is unit element of G, then $e(x, y) = x \cdot y$ is identical to the multiplication as Lie group.

Definition 5. Let (M, η, k) be a p-space. An affine connection Γ on M can be defined by

$$\Gamma(X, Y)(e) = -\frac{1}{k} X(e) \cdot Y(e) = -\frac{1}{k} \eta(e, X(e), Y(e)) \tag{2.15}$$

where X and Y are any vecter fields on M. This connection is called *the canonical connection* on M.

In local coordinates $(x_1, \dots, x_m)$ at e, this canonical connection Γ has the following expression. For $X = \partial / \partial x_i$ and $Y = \partial / \partial x_j$, we have for all $f \in F(M)$

$$(X(e) \cdot Y(e)) f = X(e) \otimes Y(e)\ f(\eta(e, x, y))$$

$$= X(e) \left(\frac{\partial f}{\partial x_k}(\eta(e, x, e)) \frac{\partial \eta_k}{\partial z_j}(e, x, e)\right)$$

$$= \frac{\partial^2 f}{\partial x_h \partial x_k}(e) \frac{\partial \eta_h}{\partial y_i}(e, e, e) \frac{\partial \eta_k}{\partial z_j}(e, e, e) + \frac{\partial f}{\partial x_k}(e) \frac{\partial^2 \eta_k}{\partial y_i \partial z_j}(e, e, e),$$

so, from (1.2), (1.3), and (2.15)

$$\Gamma(\partial / \partial x_i, \partial / \partial x_j) = -\partial^2 / \partial x_i \partial x_j - \frac{1}{k} \frac{\partial^2 \eta_k}{\partial y_i \partial z_j}(x, x, x) \partial / \partial x_k.$$

Comparing this with (2.12), we see that the Christoffel symbols Γ^k_{ij} of this connection are given by

$$\Gamma^k_{ij}(x) = -\frac{1}{k} \frac{\partial^2 \eta_k}{\partial y_i \partial z_j}(x, x, x). \tag{2.16}$$

For Example 1, this connection is the canonical connection for differentiable homogeneous systems (see [9], formula (4.3), p. 49).

For Example 2, this connection is the $(-)$-connection introduced by É. Cartan and J.A. Schouten [2]. Let M be a Lie group with unit element e. For any left invariant vector fields $\tilde{X}$ and $\tilde{Y}$ on M, from the relation (2.11), we have

$$\nabla_{\tilde{X}} \tilde{Y}(e) = X\tilde{Y} + \Gamma(X, Y) = X\tilde{Y} - X \cdot Y = 0,$$

where $X = \tilde{X}(e)$ and $Y = \tilde{Y}(e)$ (see Example 5 and Remark 6). Therefore, our connection is the $(-)$-connection (see, for instance, [3], Prop. 1.4, p. 102 and p. 104).

For Example 3, this connection is the canonical connection for symmetric spaces.

LEMMA 2. *Let M be a symmetric space. For any $e \in M$, and for all $X \in T_e(M)$,*

$$\text{i) } e \bigcirc X = -X, \qquad \text{ii) } X \bigcirc e = 2X. \tag{2.17}$$

For a proof, see 0. Loos [11], p. 76.

Now, for any symmetric space M, from Example 3, (2.15), (S_3), Proposition 4 iii), and the above Lemma 2,

$$X\cdot Y=2\Gamma(X,\ Y)=e\bigcirc(X\bigcirc Y)=(e\bigcirc X)\bigcirc(e\bigcirc Y)=(-X)\bigcirc(-Y)=X\bigcirc Y,$$

for $e\in M$, and $X,\ Y\in T_e(M)$. Therefore, this connection is the canonical connection on the symmetric space M, as is seen from the definition in 0. Loos [11], p. 83.

Example 6. Let M be a Lie group with unit element e, then the multiplication $x\bigcirc y=x\cdot y^{-1}\cdot x$ on M gives the structure of a symmetric space on M (see [11], p. 65). The canonical connection on this M is the (0)-connection introduced by É. Cartan and J. A. Schouten [2].

For any left invariant vector fields $\tilde{X}$ and $\tilde{Y}$ on M, from (2.11), Propositions 4 and 5, and Example 4,

$$\nabla_{\tilde{X}}\tilde{Y}(e)=X\tilde{Y}+\Gamma(X,\ Y)=X\cdot Y+\frac{1}{2}(-X\cdot Y-Y\cdot X)=\frac{1}{2}[X,\ Y],$$

where $X=\tilde{X}(e)$ and $Y=\tilde{Y}(e)$. This shows the above assertion (see, also, [3], Prop. 1.4, p. 102, and p. 104).

The following key lemma is essentially the same to Lemma in M. Kikkawa [10].

LEMMA 3. *Let* (M,η) *be a p-space with canonical connection. If a tensor field* S *of type* $(r,\ s)$ *on* M *satisfies the following relation* (2.18) *for all points* $x\in M$, *in local coordinates* $(x_1,\dots,x_m)$ *at* x, *where for each* x, y *runs a neighborhood of* x; *then, the covariant differentiation of* S *is zero, i.e.* $\nabla S=0$.

$$S^{i_1\cdots\cdots i_r}_{j'_1\cdots\cdots j'_s}(\eta(x,y,x))\,\frac{\partial\eta_{j'_1}}{\partial z_{j_1}}(x,y,x)\cdots\cdots\frac{\partial\eta_{j'_s}}{\partial z_{j_s}}(x,y,x).$$

$$=S^{i'_1\cdots\cdots i'_r}_{j_1\cdots\cdots j_s}(x)\,\frac{\partial\eta_{i_1}}{\partial z_{i'_1}}(x,y,x)\cdots\cdots\frac{\partial\eta_{i_r}}{\partial z_{i'_r}}(x,y,x).\tag{2.18}$$

PROOF. Operate $Y=\partial/\partial x_k$ on both sides of (2.18) for variable y, and put $y=x$. From the following formula for covariant differentiation

$$\nabla_{\partial/\partial x_k}S^{i_1\cdots\cdots i_r}_{j_1\cdots\cdots j_s}(x)=\frac{\partial}{\partial x_k}S^{i_1\cdots\cdots i_r}_{j_1\cdots\cdots j_s}(x)+\sum_{a=1}^{r}S^{i_1\cdots\cdots l\cdots\cdots i_r}_{j_1\cdots\cdots j_s}(x)\Gamma^{i_a}_{kl}(x)$$

$$-\sum_{b=1}^{s}S^{i_1\cdots\cdots i_r}_{j_1\cdots\cdots m\cdots\cdots j_s}(x)\Gamma^{m}_{kj_b}(x),\tag{2.19}$$

this lemma follows immediately, if we consider the relations (1.2), (1.3) and (2.16).

Two fundamental tensor fields on a manifold M with an affine connection are the torsion tensor field T and the curvature tensor field R. They are defined by

$$T(X,\ Y)=\nabla_XY-\nabla_YX-[X,\ Y]=\Gamma(X,\ Y)-\Gamma(Y,X),\tag{2.20}$$

$$R(X,\ Y)Z=\nabla_X\nabla_YZ-\nabla_Y\nabla_XZ-\nabla_{[X,\ Y]}Z,\tag{2.21}$$

where X, Y, and Z are any vector fields on M.

Our main aim in this paper is to prove the following theorem, by virtue of the above Lemma 3.

THEOREM 1. *Let M be a p-space with canonical connection. Then, the covariant differentiations of the torsion tensor field T and the curvature tensor field R on M are zero, i.e.* $\nabla T=0$, $\nabla R=0$. (*M is locally a so-called reductive homogeneous space in K. Nomizu* [12].)

§3. A proof of Theorem 1

The heart of our proof is the Lemma 4 below, which has many important applications.

PROPOSITION 6 (see [11], Theorem 2.6 i) a), p. 84). *Let (M, η, k) and (N, η, k) be two p-spaces with the same k, and $\phi: M \to N$ a homomorphism of M into N. Then ϕ is an affine map of M into N, for their canonical connections.*

PROOF. For any vector fields X and Y on M, we must show that

$$\phi(X(e)\cdot Y(e))=\phi(X(e))\cdot\phi(Y(e)) \qquad \text{for all } e\in M \tag{3.1}$$

(see [11], definition of affine map, p. 20). But, from our definition (2.15), this is a special case of the Lemma 4 below.

COROLLARY (see [10], §3 Remark 2). *Let (M, η) be a p-space with canonical connection. Then any displacement $\eta(e, w)$: $M \to M$ is an affine transformation of M.*

LEMMA 4 (see [11], p. 49). *The homomorphism ϕ of Proposition 6 has the following commutativity.*

$$\phi(\eta(X, Y, Z))=\eta(\phi(X), \phi(Y), \phi(Z)) \qquad \textit{for all } X, Y, Z\in T_e^{\infty}(M) \tag{3.2}$$

PROOF. For $f\in F(N)$,

$$\begin{aligned}\phi(\eta(X, Y, Z))f&=X\otimes Y\otimes Z(f\circ\phi\circ\eta(x, y, z))\\&=X\otimes Y\otimes Z(f\circ\eta(\phi(x), \phi(y), \phi(z)))\\&=\phi(X)\otimes\phi(Y)\otimes\phi(Z)\ (f\circ\eta(x', y', z'))\\&=\eta(\phi(X), \phi(Y), \phi(Z))f,\end{aligned}$$

in the above, x', y', z' denote variables on N around $\phi(e)$.

Now, let (M, η) be a p-space. Then, $\eta(x, x, x)=x$ for all $x\in M$, therefore, for higher order vector fields X, Y, and Z on M, we can define their product $\eta(X, Y, Z)$ by

$$\eta(X, Y, Z)(x)=\eta(X(x), Y(x), Z(x)), \qquad (x\in M) \tag{3.3}$$

which is also a higher order vector field on M.

Especially, $X\cdot Y$ means

$$(X\cdot Y)(x)=X(x)\cdot Y(x)=\eta(x, X(x), Y(x)), \qquad (x\in M) \tag{3.4}$$

LEMMA 5. (see [11], p. 83). *Let (M, η) be a p-space. For any higher order vector fields X, Y, Z, and any vector field W on M, we have*

$$W\eta(X, Y, Z)=\eta(WX, Y, Z)+\eta(X, WY, Z)+\eta(X, Y, WZ). \tag{3.5}$$

PROOF. For any $f \in F(M)$,

$$\begin{aligned}(W\eta(X,\ Y,\ Z)f)(w) &= W(X(w)\otimes Y(w)\otimes Z(w)f(\eta(x,\ y,\ z)))\\ &=(WX)(w)\otimes Y(w)\otimes Z(w)f(\eta(x,\ y,\ z))\\ &\quad +X(w)\otimes(WY)(w)\otimes Z(w)f(\eta(x,\ y,\ z))\\ &\quad +X(w)\otimes Y(w)\otimes(WZ)(w)\ f(\eta(x,y,z))\\ &=((\eta(WX,\ Y,\ Z)+\eta(X,\ WY,\ Z)+\eta(X,\ Y,\ WZ))f)(w).\end{aligned}$$

By a similar calculation in [11], p. 84, we obtain

PROPOSITION 7. *Let (M, η, k) be a p-space. For the canonical connection on M, the torsion tensor field T and the curvature tensor field R can be expressed by*

$$T(X,\ Y)=-\frac{1}{k}(X\cdot Y-Y\cdot X), \tag{3.6}$$

$$R(X,\ Y)Z=\frac{1}{k^2}(X\cdot(Y\cdot Z)-Y\cdot(X\cdot Z))-\frac{1}{k}(\eta(X,Y,Z)-\eta(Y,X,Z)), \tag{3.7}$$

where X, Y, and Z are any vector fields on M.

PROOF. (3.6) is obvious from the definition (2.20).

As to R, we first remark that, as a special case of (3.5),

$$X\eta(x,\ Y,\ Z)=\eta(X,\ Y,\ Z)+\eta(x,\ XY,\ Z)+\eta(x,\ Y,\ XZ), \qquad (x\in M)$$

i.e.

$$X(Y\cdot Z)=\eta(X,Y,Z)+(XY)\cdot Z+Y\cdot(XZ). \tag{3.8}$$

From (2.21), (2.11), and the above (3.8),

$$\begin{aligned}R(X,\ Y)Z &= \nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z\\ &=X(YZ-\frac{1}{k}Y\cdot Z)-\frac{1}{k}X\cdot(YZ-\frac{1}{k}Y\cdot Z)\\ &\quad -Y(XZ-\frac{1}{k}X\cdot Z)+\frac{1}{k}Y\cdot(XZ-\frac{1}{k}X\cdot Z)\\ &\quad -(XY-YX)Z+\frac{1}{k}(XY-YX)\cdot Z\\ &=\frac{1}{k^2}X\cdot(Y\cdot Z)-\frac{1}{k^2}Y\cdot(X\cdot Z)-\frac{1}{k}\eta(X,\ Y,\ Z)+\frac{1}{k}\eta(Y,\ X,\ Z).\end{aligned}$$

Combining this Proposition 7 and Lemma 4, we gain a proof of the following

PROPOSITION 8. *Let (M, η, k) and (N, η, k) be two p-spaces with the same k, and let $\phi: M\to N$ be a homomorphism of M into N. Then, the torsion tensor fields T and the curvature tensor fields R on M and N (here the same symbols are used) for their canonical connections, satisfy the following relations.*

$$\phi(T(X,\ Y))=T(\phi(X),\phi(Y)), \tag{3.9}$$

$$\phi(R(X, Y)Z)=R(\phi(X), \phi(Y))\phi(Z), \tag{3.10}$$

for any vector fields X, Y, *and* Z *on* M.

Remark 7. The above Proposition 8 holds, generally, for any manifolds M and N with affine connection and for any affine map $\phi: M \to N$ (see [11], p. 28). See, also, Remark 8 below.

T and R are expressed, in local coordinates $(x_1,\dots,x_m)$ on M, by

$$T(\partial/\partial x_i, \partial/\partial x_j)=T^k_{ij}(x)\partial/\partial x_k, \tag{3.11}$$

$$R(\partial/\partial x_i, \partial/\partial x_j)\partial/\partial x_k=R^h_{kij}(x)\partial/\partial x_h. \tag{3.12}$$

we have similar expressions for T and R on N, in local coordinates $(y_1,\dots,y_n)$ on N.

Using these expressions, (3.9) and (3.10) can be written explicitly, on account of (2.3), in the following forms.

$$T^r_{pq}(\phi(x))\frac{\partial \phi_p}{\partial x_i}(x)\frac{\partial \phi_q}{\partial x_j}(x)=T^k_{ij}(x)\frac{\partial \phi_r}{\partial x_k}(x), \tag{3.9$'$}$$

$$R^s_{rpq}(\phi(x))\frac{\partial \phi_r}{\partial x_k}(x)\frac{\partial \phi_p}{\partial x_i}(x)\frac{\partial \phi_q}{\partial x_j}(x)=R^h_{kij}(x)\frac{\partial \phi_s}{\partial x_h}(x). \tag{3.10$'$}$$

Therefore, we have finally a proof of Theorem 1.

PROOF. The displacement $\phi=\eta(e, w)$: $M \to M$ is an automorphism of M, and

$$\frac{\partial \phi_i}{\partial x_j}(x)=\frac{\partial \eta_i}{\partial z_j}(e, w, x).$$

Putting $x=e$, the desired relations (2.18) follow, immediately, from (3.9)$'$ and (3.10)$'$.

Remark 8. For any tensor field S on M, which is expressed by η, we can show that $\nabla S=0$. The reason is that the heart of our proof is the Lemma 4, as already pointed out.

§4. The tangent algebra

Observing the tangent algebras of locally reductive spaces, K. Yamaguti has introduced in [13] an algebraic system, called general Lie triple system (afterward Lie triple algebra by M. Kikkawa in [6]), which has important and leading significance in our investigation.

Definition 6. Let V be a finite dimensional vector space (over a field). A bilinear composition $[X, Y]$, and a trilinear composition $\langle X, Y, Z\rangle$ are defined in V, and satisfy the following conditions, then V is called a *Lie triple algebra* (or a *general Lie triple system*).

(T_1) $[X, X]=0$,

(T_2) $\langle X, X, Y\rangle=0$,

(T_3) $[[X, Y], Z]+[[Y, Z], X]+[[Z, X], Y]+\langle X, Y, Z\rangle+\langle Y, Z, X\rangle+\langle Z, X, Y\rangle=0$,

(T_4) $\langle [X, Y], Z, W\rangle+\langle [Y, Z], X, W\rangle+\langle [Z, X], Y, W\rangle=0$,

(T_5) $\langle X, Y, [Z, W]\rangle=[\langle X, Y, Z\rangle, W]+[Z, \langle X, Y, W\rangle]$,

(T_6) $\langle X, Y\langle U, V, W\rangle\rangle$
$=\langle\langle X, Y, U\rangle, V, W\rangle+\langle U, \langle X, Y, V\rangle, W\rangle+\langle U, V, \langle X, Y, W\rangle\rangle$.

Remark 9. If the trilinear composition in the above definition vanishes identically then this definition becomes that of Lie algebra, and if the bilinear composition in the above definition vanishes identically then this definition becomes that of Lie triple system.

THEOREM 2 (K. Yamaguti [13] also see M.Kikkawa [5]). *Let M be a manifold with an affine connection. If the covariant differentiations of the torsion tensor field T and the curvature tensor field R on M are zero, i.e. $\nabla T=0$, and $\nabla R=0$, then for any point $e\in M$, the tangent space $T_e(M)$ has the structure of a Lie triple algebra by the following compositions.*

$$[X, Y]=-T(X, Y), \tag{4.1}$$

$$\langle X, Y, Z\rangle=-R(X, Y)Z. \tag{4.2}$$

PROOF. (T_3) and (T_4) are Bianchi's identities, and (T_5) and (T_6) are Ricci's identities.

PROPOSITION 9. *Let (M, η, k) be a p-space with canonical connection. The trilinear composition in the tangent Lie triple algebra $T_e(M)$ $(e\in M)$*, has the following expressions.

$$\langle X, Y, Z\rangle=-\frac{1}{k^2}(X\cdot(Y\cdot Z)-Y\cdot(X\cdot Z)) \\ +\frac{1}{k}(\eta(X, Y, Z)-\eta(Y, X, Z)), \tag{4.3}$$

$$\langle X, Y, Z\rangle=\frac{1}{k^2}(X\cdot(Y\cdot Z)-Y\cdot(X\cdot Z)) \\ -\frac{1}{k^2}((X\cdot Y-Y\cdot X)\cdot Z), \tag{4.4}$$

$$\langle X, Y, Z\rangle=\frac{1}{2k}(\eta(X, Y, Z)-\eta(Y, X, Z)) \\ -\frac{1}{2k^2}((X\cdot Y-Y\cdot X)\cdot Z). \tag{4.5}$$

Remark 10. From the expression (4.4), we can directly show the condition (T_3) for the tangent Lie triple algebra $T_e(M)$.

Remark 11. The expression (4.3) corresponds to the formula (3.6) in [10], and the expression (4.4) corresponds to the formula (4.9) in [9].

PROOF. From (4.2), (4.3) is reduced to (3.7).

Now, consider the following triple multiplication ϕ on M.

$$\begin{aligned}\phi(x, y, z)&=\eta(\eta(e, x, e), y, \eta(e, x, z)) \\ &=\eta(e, x, \eta(e, \eta(x, e, y), z)).\end{aligned}$$

For $X, Y, Z\in T_e(M)$, by virtue of Propositions 4 and 5

$$\begin{aligned}\phi(X, Y, Z)&=\eta(\eta(e, X, e), Y, \eta(e, e, Z))+\eta(\eta(e, e, e), Y, \eta(e, X, Z)) \\ &=k\eta(X, Y, Z)+Y\cdot(X\cdot Z).\end{aligned}$$

On the other hand, also using Corollary of Proposition 5

$$\phi(X, Y, Z) = \eta(e, X, \eta(e, \eta(e, e, Y), Z)) + \eta(e, e, \eta(e, \eta(X, e, Y), Z))$$
$$= X\cdot(Y\cdot Z) - (X\cdot Y)\cdot Z.$$

Comparing these relations, and using (4.3), we obtain (4.4).

If (4.3) and (4.4) are added side by side, (4.5) follows immediately.

COROLLARY 1. *If M denotes the p-space of a Lie group G, then the tangent Lie triple algebra $T_e(M)$ is identical to the Lie algebra of G.*

PROOF. From Example 2 and Corollary of Proposition 6, we may assume e to be unit element of G. $\langle X, Y, Z\rangle = 0$ follows from (4.4) and Proposition 4 iii), and

$$[X, Y] = -T(X, Y) = X\cdot Y - Y\cdot X$$

from (3.6), therefore Example 5 shows Corollary 1. (see Remark 9 and Remark 12 below.)

COROLLARY 2. *If $(M, \eta, -2)$ denotes the p-space of a symmetric space S, then the tangent Lie triple algebra $T_e(M)$ is identical to the Lie triple system of S.*

PROOF. For the multiplication on S, we have

$$x\bigcirc y = S_x y = \eta(x, y, y),$$

therefore, by virtue of Proposition 5

$$X\bigcirc Y = \eta(X, Y, e) + \eta(X, e, Y) \qquad \text{for } X, Y\in T_e(M).$$

Also, we know $X\bigcirc Y = X\cdot Y$ (see p. 10), so using (2.10)

$$2X\cdot Y = \eta(X, Y, e) \qquad \text{for } X, Y\in T_e(M) \tag{4.6}$$

For vector fields X, and Y on M, making use of Lemma 2, we see

$$\eta(x, Y, x) = x\bigcirc(Y\bigcirc x) = -2Y, \qquad (x\in M)$$

so, appling Lemma 5 we have

$$\eta(X, Y, x) + \eta(x, XY, x) + \eta(x, Y, X) = -2XY. \tag{4.7}$$

If $X = \partial/\partial x_i$, and $Y = \partial/\partial x_j$ (locally), then $XY = YX$, therefore (4.7) and the similar relation interchanged X and Y lead to the following commutativity, with the aid of (4.6)

$$X\cdot Y = Y\cdot X, \tag{4.8}$$

which holds for all vector fields X, and Y on M, by linearity. So, the torsion tensor field T on M is zero identically.

Finally, from (4.5) and the above (4.8)

$$\langle X, Y, Z\rangle = -\frac{1}{4}(\eta(X, Y, Z) - \eta(Y, X, Z))$$

$$= -\frac{1}{4}(X\bigcirc(Y\bigcirc Z) - Y\bigcirc(X\bigcirc Z)) \qquad \text{for } X, Y, Z\in T_e(M), \tag{4.9}$$

which is equal to the trilinear composition $[X, Y, Z]$ in 0. Loos [11], Lemma 2.4, p. 80.

Remark 12. The canonical connections on M and S, as p-space and symmetric space, are the same; therefore, the above Corollary 2 is obvious from 0. Loos [11], Lemma 2.4, p. 80, and Prop. 2.5, p. 83. Our above consideration gives another proof for this fact, from a general point of view of p-spaces (see also Remark 11).

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