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# *On the Cohomology of Certain Quotient Manifolds of the Real Stiefel Manifolds and Their Applications*

Tsutomu Yasui

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## § 0. Introduction

Throughout this paper, $p$ will denote an odd prime integer.

Let $S^{2n+1}$ be the unit $(2n+1)$-sphere in the complex $(n+1)$-space. Then the free actions of $S^1=\{e^{i\theta}\,|\,0\leq\theta<2\pi\}$ and $Z_p=\{e^{i\theta}\,|\,\theta=2\pi h/p,\ h=0,\cdots,p-1\}$ on $S^{2n+1}$ are defined by $e^{i\theta}(z_0,\cdots,z_n)=(e^{i\theta}z_0,\cdots,e^{i\theta}z_n)$.

Let $V_{2n,k}$ be the Stiefel manifold of orthonormal $k$-frames in the real $2n$-space $R^{2n}$. We define free actions of $S^1$ and $Z_p$ on $V_{2n,k}$ such that $e^{i\theta}$ operates on each vector of $k$-frame as above. We consider the quotient manifolds

$$Z_{2n,k}=V_{2n,k}/S^1, \qquad X_{2n,k}=V_{2n,k}/Z_p.$$

Then $Z_{2n,1}=CP^{n-1}$, the real $2n-2$ dimensional complex projective space, and $X_{2n,1}=L^{n-1}(p)$, the $2n-1$ dimensional mod $p$ lens space.

Let $\xi$ and $\eta$ be the canonical complex line bundles over $CP^\infty$ and $L^\infty(p)$, respectively. Then the above manifolds $Z_{2n,k}$ and $X_{2n,k}$ are homotopy equivalent to the total spaces of the associated $V_{2n,k}$-bundles of $n\xi$ and $n\eta$, respectively, as is shown in Proposition 1.3. Consequently, it is expected that the cohomology structures of $Z_{2n,k}$ and $X_{2n,k}$ give us the informations about the structures of $n\xi$ and $n\eta$ and so the immersion problem for lens spaces $L^n(p)$.

Recently, S. Gitler and D. Handel [5] have considered the projective Stiefel manifolds, which are the above manifolds $X_{n,k}$ for $p=2$ (in this case, $n$ need not be even), and determined their mod 2 cohomology algebras and the actions of the Steenrod squares up to a small indeterminancy. Also, P. F. Baum and W. Browder [1] have determined completely the actions of the Steenrod squares when $n$ is a power of 2. Moreover S. Gitler [6] has applied these results to the immersion problem for the real projective spaces.

The purpose of this paper is to study the mod $p$ cohomology structures of $Z_{2n,k}$ and $X_{2n,k}$ and to apply these results to the problems of independent cross sections of $n\eta$ and immersions of $L^n(p)$.

In §1, we prove Theorem 1.11, which determines the mod $p$ cohomology algebras $H^*(Z_{2n,k})$ and $H^*(X_{2n,k})$. Furthermore the generators are given in

Theorem 2.7, using the universal Pontrjagin classes $p_j$ and Euler class $\chi$, which is proved by the analogous method in [5]. The mod $p$ reduced power operations $\mathcal{P}^i$ in these algebras are studied in §3, using Theorem 2.7 and the well-known results on $\mathcal{P}^i p_j$ and $\mathcal{P}^i \chi$. Also we study the Bockstein homomorphism $\beta$ in §4, using the results of [1, MAIN THEOREM I (7.12)]. $\mathcal{P}^i$ and $\beta$ are determined explicitly in Theorems 3.10–11 and 4.12 for $n=n'p^r$ $(r\geq 1)$ and some $k$.

For the applications, we study the relations between $Z_{2n,k}$ and $Z_{2n+2m,k}$ in §5 and prove Proposition 6.4. Finally, we apply Proposition 6.4 to Theorem 6.2 which is a non-existance theorem of $k$ independent cross sections of the bundle $m\eta$ over $L^n(p)$. By Theorem 6.2 and T. Kobayashi's Theorem [7, Theorem 1], we obtain Theorem 6.3, which is a non-immersion theorem for lens spaces $L^n(p)$.

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## § 1. The mod $p$ cohomology of $X_{2n,k}$ and $Z_{2n,k}$

In this paper, the cohomology $H^*(\ \ )$ will be understood to have $Z_p$ for coefficients, unless otherwise stated.

Let $V_{2n,k}$ be the Stiefel manifold of orthonormal $k$-frames in the real $2n$-space $R^{2n}$ and define a free action of $S^1=\{e^{i\theta}\,|\,0\leq\theta<2\pi\}$ on $V_{2n,k}$ by considering

$$e^{i\theta}=\begin{pmatrix} e^{i\theta} & & 0 \\ & \ddots & \\ 0 & & e^{i\theta} \end{pmatrix}\in U(n)\subset SO(2n).$$

We consider the following quotient manifolds:

$$Z_{2n,k}=V_{2n,k}/S^1, \qquad X_{2n,k}=V_{2n,k}/Z_p$$

where $Z_p=\{e^{i\theta}\,|\,\theta=2\pi h/p,\ h=0,1,\cdots,p-1\}\subset S^1$.

Let $\xi$ and $\eta$ be the canonical complex line bundles over the infinite dimensional complex projective space $CP^\infty$ and the mod $p$ lens space $L^\infty(p)$, respectively, and $n\xi$ (resp. $n\eta$) the Whitney sum of $n$ copies of $\xi$ (resp. $\eta$). The real restriction of $n\xi$ (resp. $n\eta$) is denoted by the same notation $n\xi$ (resp. $n\eta$). The associated $V_{2n,k}$-bundles of $n\xi$ and $n\eta$ are the following:

$$V_{2n,k}\longrightarrow S^\infty\times_{S^1}V_{2n,k}\longrightarrow CP^\infty, \tag{1.1}$$

$$V_{2n,k}\longrightarrow S^\infty\times_{Z_p}V_{2n,k}\longrightarrow L^\infty(p), \tag{1.2}$$

where $S^\infty$ is the infinite dimensional sphere and the projections are defined by the natural projections $S^{\infty}\longrightarrow S^{\infty}/S^1=CP^{\infty}$ and $S^{\infty}\longrightarrow S^{\infty}/Z_p=L^{\infty}(p)$, respectively.

PROPOSITION 1.3. *The manifolds* $Z_{2n,k}$ *(resp.* $X_{2n,k}$*) and* $S^{\infty}\times_{S^1}V_{2n,k}$ *(resp.* $S^{\infty}\times_{Z_p}V_{2n,k}$*) are of the same homotopy type and the natural projection* $V_{2n,k}\longrightarrow Z_{2n,k}$ *(resp.* $V_{2n,k}\longrightarrow X_{2n,k}$*) can be identified with the inclusion.*

PROOF. The following diagram is commutative:

$$\begin{array}{ccccc} S^1 & \longrightarrow & S^{\infty}\times V_{2n,k} & \longrightarrow & S^{\infty}\times_{S^1}V_{2n,k} \\ \| & & \downarrow & & \downarrow \\ S^1 & \longrightarrow & V_{2n,k} & \longrightarrow & Z_{2n,k}, \end{array} \tag{1.4}$$

where vertical maps are the projections. The projection $S^{\infty}\times V_{2n,k}\longrightarrow V_{2n,k}$ is obviously a homotopy equivalence and the inclusion map $V_{2n,k}\longrightarrow S^{\infty}\times V_{2n,k}$ is its homotopy inverse. Hence, by the homotopy exact sequences of the fibrations and the five lemma, the projection $S^{\infty}\times_{S^1}V_{2n,k}\longrightarrow Z_{2n,k}$ induces isomorphisms of all homotopy groups, and we obtain $S^{\infty}\times_{S^1}V_{2n,k}\simeq Z_{2n,k}$. Similarly it follows that $S^{\infty}\times_{Z_p}V_{2n,k}\simeq X_{2n,k}$. Q.E.D.

According to Proposition 1.3, we identify the space $S^{\infty}\times_{S^1}V_{2n,k}$ with $Z_{2n,k}$ and $S^{\infty}\times_{Z_p}V_{2n,k}$ with $X_{2n,k}$.

Now, let $f_n\colon CP^{\infty}\longrightarrow BSO(2n)$ be a classifying map of $n\xi$. Then $f_n\pi$ is a classifying map of $n\eta$, since $\eta=\pi^*\xi$, where $\pi\colon L^{\infty}(p)\longrightarrow CP^{\infty}$ is the natural projection. Therefore we obtain the following homotopy commutative diagram:

$$\begin{array}{ccccc} V_{2n,k} & = & V_{2n,k} & = & V_{2n,k} \\ {\scriptstyle i}\downarrow & & {\scriptstyle i}\downarrow & & \downarrow \\ X_{2n,k} & \xrightarrow{\tilde{\pi}} & Z_{2n,k} & \xrightarrow{\tilde{f}_n} & BSO(2n-k) \\ {\scriptstyle p}\downarrow & & {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi'} \\ L^{\infty}(p) & \xrightarrow{\pi} & CP^{\infty} & \xrightarrow{f_n} & BSO(2n). \end{array} \tag{1.5}$$

The mod $p$ cohomology structures of $V_{2n,k}$ and $BSO(n)$ are the following ([2], [3] and [9, Theorem 32]):

$$H^*(V_{2n,k})=\begin{cases} \wedge(v_{n-k'+1},\cdots,v_{n-1},v) & \text{if}\quad k=2k'-1 \\ \wedge(v_{n-k'+1},\cdots,v_{n-1},v,v') & \text{if}\quad k=2k', \end{cases} \tag{1.6}$$

where $\deg v_j=4j-1$, $\deg v=2n-1$ and $\deg v'=2n-k$.

$$H^*(BSO(n))=\begin{cases} Z_p[p_1,\cdots,p_{n'-1},\chi] & \text{if}\quad n=2n' \\ Z_p[p_1,\cdots,p_{n'-1}] & \text{if}\quad n=2n'-1, \end{cases} \tag{1.7}$$

where $p_j$ is the $j$-th Pontrjagin class of the universal oriented $n$-plane bundle, $\chi$ is its Euler class. Notice that $\chi^2=p_{n'}$, for $n=2n'$. Moreover the elements $v_j$ and $v$ are transgressive in the fibration $V_{2n,k}\longrightarrow BSO(2n-k)\longrightarrow BSO(2n)$, and

(1.8) $$\tau v_j = p_j, \qquad \tau v = \chi.$$

Also, it is well-known that

(1.9) $$H^*(CP^\infty) = Z_p[y], \quad \text{where deg } y = 2,$$

(1.10) $$H^*(L^\infty(p)) = \wedge(x) \otimes Z_p[y],$$

where deg $x=1$, deg $y=2$ and $\beta x = y$ ($\beta$ denotes the Bockstein homomorphism).

THEOREM 1.11. *Suppose* $0<k<2n$ *and set* $k'=[(k+1)/2]$. *Let*

$$N = N(n, k) = \min\left\{4i \mid n-k'+1 \leq i \leq n-1, \binom{n}{i} \not\equiv 0 \bmod p\right\}.$$

*Then the* mod $p$ *cohomology algebras of* $X_{2n,k}$ *and* $Z_{2n,k}$ *are as follows*:

(*a*) *If* $N$ *does not exist or if* $N$ *exists and* $2n<N$,

(1.11.1) $$H^*(Z_{2n,k}) = \begin{cases} \wedge(z_{n-k'+1}, \cdots, z_{n-1}) \otimes Z_p[y]/(y^n) & \text{for odd } k \\ \vee(z_{n-k'+1}, \cdots, z_{n-1}, z') \otimes Z_p[y]/(y^n) & \text{for even } k, \end{cases}$$

(1.11.2) $$H^*(X_{2n,k}) = \begin{cases} \wedge(z_{n-k'+1}, \cdots, z_{n-1}) \otimes \wedge(x) \otimes Z_p[y]/(y^n) & \text{for odd } k \\ \vee(z_{n-k'+1}, \cdots, z_{n-1}, z') \otimes \wedge(x) \otimes Z_p[y]/(y^n) & \text{for even } k. \end{cases}$$

(*b*) *If* $N$ *exists and* $2n=N=4i_0$,

(1.11.3) $$H^*(Z_{2n,k}) = \begin{cases} \wedge(z_{n-k'+1}, \cdots, \hat{z}_{i_0}, \cdots, z_{n-1}) \otimes Z_p[y]/(y^n) \otimes \wedge(\bar{z}_{i_0}) \\ \qquad\qquad \text{for odd } k \\ \vee(z_{n-k'+1}, \cdots, \hat{z}_{i_0}, \cdots, z_{n-1}, z') \otimes Z_p[y]/(y^n) \otimes \wedge(\bar{z}_{i_0}) \\ \qquad\qquad \text{for even } k, \end{cases}$$

(1.11.4) $$H^*(X_{2n,k}) = \begin{cases} \wedge(z_{n-k'+1}, \cdots, \hat{z}_{i_0}, \cdots, z_{n-1}) \\ \qquad \otimes \wedge(x) \otimes Z_p[y]/(y^n) \otimes \wedge(\bar{z}_{i_0}) \quad \text{for odd } k \\ \vee(z_{n-k'+1}, \cdots, \hat{z}_{i_0}, \cdots, z_{n-1}, z') \\ \qquad \otimes \wedge(x) \otimes Z_p[y]/(y^n) \otimes \wedge(\bar{z}_{i_0}) \quad \text{for even } k. \end{cases}$$

(*c*) *If* $N$ *exists and* $2n>N=4i_0$,

(1.11.5) $$H^*(Z_{2n,k}) = \begin{cases} \wedge(z_{n-k'+1}, \cdots, \hat{z}_{i_0}, \cdots, z_{n-1}, z) \otimes Z_p[y]/(y^{2i_0}) \\ \qquad\qquad \text{for odd } k \end{cases}$$

$$\begin{cases} \vee(z_{n-k'+1}, \cdots, \hat{z}_{i_0}, \cdots, z_{n-1}, z, z')\otimes Z_p[y]/(y^{2i_0}) & \textit{for even } k, \end{cases}$$

$$(1.11.6)\qquad H^*(X_{2n,k})=\begin{cases} \wedge(z_{n-k'+1}, \cdots, \hat{z}_{i_0}, \cdots, z_{n-1}, z)\otimes\wedge(x)\otimes Z_p[y]/(y^{2i_0}) & \textit{for odd } k \\ \vee(z_{n-k'+1}, \cdots, \hat{z}_{i_0}, \cdots, z_{n-1}, z, z')\otimes\wedge(x)\otimes Z_p[y]/(y^{2i_0}) & \textit{for even } k. \end{cases}$$

*Here* deg $z_j=4j-1$, deg $\bar{z}_{i_0}=4i_0-1$, deg $z=2n-1$, deg $z'=2n-k$, deg $x=1$, deg $y=2$ *and* $\vee(h_1, \cdots, h_s)$ *means the algebra with* $h_1, \cdots, h_s$ *as the simple system of generators, and* $\hat{z}_{i_0}$ *indicates that* $z_{i_0}$ *has been omitted. Moreover, we have the following relations*:

$$(1.11.7)\qquad \begin{cases} i^*z_j=v_j, \quad i^*\bar{z}_{i_0}=v_{i_0}-\binom{n}{i_0}v, \quad i^*z=v, \quad i^*z'=v'; \\ p^*x=x, \quad p^*y=y; \quad z'^2=\binom{n}{n-k'}y^{2n-k}, \end{cases}$$

*where* $p$ *and* $i$ *are the maps in* (1.5).

Remark 1.12. When $n=p^r$ or $2p^r$ $(r\geq 1)$, the case $(c)$ does not appear and $\vee(\cdots)$ are $\wedge(\cdots)$. In fact, $N(p^r, k)$ does not exist for any $k$, and $N(2p^r, k)=4p^r$ if $k'>p^r$ and $N(2p^r, k)$ does not exist if $k'\leq p^r$. Moreover $z'^2=0$, since $y^{2n-k}=y^n=0$ if $n=2p^r=k$, and $\binom{n}{n-k'}\equiv 0 \bmod p$ otherwise.

Remark 1.13. By (2.7.3) of Theorem 2.7, the element $\bar{z}_{i_0}$ will be denoted simply by $z_{i_0}$ in §§3–5.

Proof of Theorem 1.11. We shall prove (1.11.3) and the others are proved similarly. Let $\{E_r, d_r\}$ be the mod $p$ cohomology spectral sequence of the bundle (1.1). Since $n\xi$ is orientable, the local system of the bundle (1.1) is trivial and we have $E_2=H^*(V_{2n,k})\otimes H^*(CP^\infty)$.

If $k$ is odd, $E_2=\wedge(v_{n-k'+1}, \cdots, v_{n-1}, v)\otimes Z_p[y]$. From (1.8) and the naturality of the transgression, we have

$$(1.14)\qquad \tau v_j=p_j(n\xi)=\binom{n}{j}y^{2j}, \quad \tau v=\chi(n\xi)=y^n.$$

Hence, the first non-zero differential is $d_{2n}=d_{4i_0}$ and

$$d_{2n}v_{i_0}=\binom{n}{i_0}y^{2i_0},\ d_{2n}v=y^n,\ d_{2n}v_j=0\ \ (j=n-k'+1, \cdots, \hat{i}_0, \cdots, n-1),$$

$$E_{2n+1}=\wedge(v_{n-k'+1}, \cdots, \hat{v}_{i_0}, \cdots, v_{n-1})\otimes\wedge\left(v_{i_0}-\binom{n}{i_0}v\right)\otimes Z_p[y]/(y^n).$$

Since $y^n=0$ in $E_{2n+1}$, we have $d_r=0$ for $r\geq 2n+1$ and $E_{2n+1}=E_\infty$. Therefore we have (1.11.3) and (1.11.7) by [3, Proposition 7.4].

If $k$ is even, $E_2=\wedge(v_{n-k'+1}, \cdots, v_{n-1}, v, v')\otimes Z_p[y]$ and

$$E_\infty=\wedge(v_{n-k'+1}, \cdots, \hat{v}_{i_0}, \cdots, v_{n-1}, v')\otimes\wedge\left(v_{i_0}-\binom{n}{i_0}v\right)\otimes Z_p[y]/(y^n),$$

similarly. Now let $\{E'_r, d'_r\}$ be the mod $p$ cohomology spectral sequence of the fibration $V_{2n,k}\longrightarrow BSO(2n-k)\xrightarrow{\pi'}BSO(2n)$, then we have $E'_\infty=\wedge(v')\otimes Z_p[p_1, \cdots, p_{n-k'}]$ by (1.8). The map $\tilde{f}_n$ in (1.5) induces $\tilde{f}_n^*\colon \{E'_r, d'_r\}\longrightarrow\{E_r, d_r\}$ such that $\tilde{f}_n^*=1\otimes f_n^*\colon E'_2\longrightarrow E_2$ and $\tilde{f}_n^*v'=v'$, $\tilde{f}_n^*p_j=\binom{n}{j}y^{2j}$ for $\tilde{f}_n^*\colon E'_\infty\longrightarrow E_\infty$. The element $v'\in E'_\infty$ is the image of $\chi'\in H^*(BSO(2n-k))=Z_p[p_1, \cdots, p_{n-k'-1}, \chi']$ by the projection $H^*(BSO(2n-k))\longrightarrow\sum_{t=0}^{\infty}E'^{0,t}_\infty=\wedge(v')$. Therefore the element $v'\in E_\infty$ is the image of $z'=\tilde{f}_n^*\chi'\in H^*(Z_{2n,k})$ by the projection $H^*(Z_{2n,k})\longrightarrow\sum_{t=0}^{\infty}E^{0,t}_\infty$. These facts and [2, Proposition 8.1 (b)] imply (1.11.3). Since $\pi'^*p_{n-k'}=\chi'^2$ by (1.7), we have

$$z'^2=\tilde{f}_n^*\chi'^2=\tilde{f}_n^*\pi'^*p_{n-k'}=p^*f_n^*p_{n-k'}=\binom{n}{n-k'}y^{2n-k}. \qquad \text{Q.E.D.}$$

Now, we study the homomorphism in cohomology induced by the projection $\tilde{\pi}\colon X_{2n,k}\longrightarrow Z_{2n,k}$ in (1.5).

LEMMA 1.15. *The homomorphism $\tilde{\pi}^*\colon H^*(Z_{2n,k})\longrightarrow H^*(X_{2n,k})$ is a monomorphism and $\tilde{\pi}^*y=y$. Moreover, we can choose the classes $z_j$, $\bar{z}_{i_0}$, $z$ and $z'$ such that $\tilde{\pi}^*z_j=z_j$, $\tilde{\pi}^*\bar{z}_{i_0}=\bar{z}_{i_0}$, $\tilde{\pi}^*z=z$ and $\tilde{\pi}^*z'=z'$.*

PROOF. Consider the following commutative diagram:

$$\begin{array}{ccccc} S^1 & \xrightarrow{i} & X_{2n,k} & \xrightarrow{\tilde{\pi}} & Z_{2n,k} \\ \| & & p\downarrow & & p\downarrow \\ S^1 & \xrightarrow{i} & L^\infty(p) & \xrightarrow{\pi} & CP^\infty. \end{array}$$

The homomorphism $i^*\colon H^*(L^\infty(p))\longrightarrow H^*(S^1)$ is an epimorphism and so it follows that $i^*\colon H^*(X_{2n,k})\longrightarrow H^*(S^1)$ is an epimorphism. Therefore each differential is trivial in the spectral sequence of the fibration $S^1\longrightarrow X_{2n,k}\xrightarrow{\tilde{\pi}}Z_{2n,k}$ and the homomorphism $\tilde{\pi}^*$ is a monomorphism. Q.E.D.

By this lemma, it is sufficient to consider the structure of $H^*(Z_{2n,k})$ for studying that of $H^*(X_{2n,k})$.

## § 2. The mod $p$ cohomology of $X_{2n,k}$ and $Z_{2n,k}$ (continued)

We study the homomorphisms induced by the projections $\nu_k\colon Z_{2n,k} \longrightarrow Z_{2n,k-1}$ and $\nu_k\colon X_{2n,k} \longrightarrow X_{2n,k-1}$ when $k$ is even.

We notice that, if one of $N(n, 2k')$ and $N(n, 2k'-1)$ of Theorem 1.11 exists, then the other exists and they are equal.

LEMMA 2.1. *Let $k=2k'$. Then $\nu_k^*\colon H^*(Z_{2n,k-1}) \longrightarrow H^*(Z_{2n,k})$ and $\nu_k^*\colon H^*(X_{2n,k-1}) \longrightarrow H^*(X_{2n,k})$ are both monomorphic. Moreover*

$$\nu_k^* z_j = z_j, \quad \nu_k^* \bar{z}_{i_0} = \bar{z}_{i_0}, \quad \nu_k^* z = z, \quad \nu_k^* x = x, \quad \nu_k^* y = y.$$

PROOF. Consider the following homotopy commutative diagram:

$$\begin{array}{ccccc} S^{2n-k} & \longrightarrow & V_{2n,k} & \xrightarrow{\nu_k} & V_{2n,k-1} \\ \| & & i\downarrow & & i\downarrow \\ S^{2n-k} & \longrightarrow & Z_{2n,k} & \xrightarrow{\nu_k} & Z_{2n,k-1}. \end{array}$$

Then the lemma is proved similarly as Lemma 1.15. Q.E.D.

If $k=2k'-1$, we obtain the following short exact sequence:

(2.2)
$$0 \longrightarrow H^*(BSO(2n), BSO(2n-k)) \xrightarrow{j^*} H^*(BSO(2n)) \xrightarrow{\pi'^*} H^*(BSO(2n-k)) \longrightarrow 0.$$

Since $\pi'^* p_j = 0$ for $n-k'+1 \leq j \leq n-1$, and $\pi'^* \chi = 0$, there exist unique classes $U_j$ $(n-k'+1 \leq j \leq n-1)$ and $U$ in $H^*(BSO(2n), BSO(2n-k))$ such that

$$j^* U_j = p_j \quad (j = n-k'+1, \dots, n-1), \qquad j^* U = \chi. \tag{2.3}$$

By the mapping cylinder considerations in the diagram (1.5), we have the following homotopy commutative diagram:

(2.4)
$$\begin{array}{ccccc} & & (CV_{2n,k}, V_{2n,k}) & & \\ & \swarrow & h\downarrow & \searrow g & \\ (L^\infty(p), X_{2n,k}) & \xrightarrow{\pi} & (CP^\infty, Z_{2n,k}) & \xrightarrow{\bar{f}_n} & (BSO(2n), BSO(2n-k)) \end{array}$$

where $CV_{2n,k}$ is the cone over $V_{2n,k}$.

LEMMA 2.5. *Let $0<k<2n$ and $k=2k'-1$. Then $g^* U_j = \delta_1 v_j$ for $n-k'+1 \leq j \leq n-1$ and $g^* U = \delta_1 v$, where $g$ is the map of (2.4) and $\delta_1\colon H^{*-1}(V_{2n,k}) \xrightarrow{\approx} H^*(CV_{2n,k}, V_{2n,k})$.*

PROOF. According to [8, Lemma 5.1], the following diagram is commutative:

$$\begin{tikzcd}
 & H^*(BSO(2n),\, BSO(2n-k)) \arrow[dl, "j^*"'] \arrow[d] \arrow[dr, "g^*"] & \\
H^*(BSO(2n)) & H^*(BSO(2n),\, V_{2n,k}) \arrow[l] \arrow[r] \arrow[d] \arrow[dl] & H^*(CV_{2n,k},\, V_{2n,k}) \\
H^*(BSO(2n),\, *) \arrow[u, "="] & H^*(BSO(2n-k),\, V_{2n,k}) \arrow[l] & H^{*-1}(V_{2n,k}) \arrow[u, "\delta_1", "\approx"'] \arrow[ul, "\delta"'] \arrow[l, "\delta"]
\end{tikzcd}$$

Since $\tau v_j = p_j$, we have $p_j \in j^* g^{*-1}\delta_1 v_j$. On the other hand $j^* U_j = p_j$ and $j^*$ is a monomorphism, and so $U_j \in g^{*-1}\delta_1 v_j$. Therefore $g^* U_j = \delta_1 v_j$. Similarly we have $g^* U = \delta_1 v$. Q.E.D.

By the diagram (2.4), we obtain the following commutative diagram of the exact sequences for odd $k$:

$$\begin{tikzcd}
 & 0 \arrow[r] & H^*(BSO(2n),\, BSO(2n-k)) \arrow[r, "j^*"] \arrow[d, "\bar f_n^*"] & H^*(BSO(2n)) \arrow[r, "\pi'^*"] \arrow[d, "f_n^*"] & H^*(BSO(2n-k)) \arrow[r] \arrow[d, "\tilde f_n^*"] & 0 \\
\cdots \arrow[r] & H^{*-1}(Z_{2n,k}) \arrow[r, "\delta"] & H^*(CP^\infty,\, Z_{2n,k}) \arrow[r, "i^*"] & H^*(CP^\infty) \arrow[r, "p^*"] & H^*(Z_{2n,k}) \arrow[r] & \cdots.
\end{tikzcd} \tag{2.6}$$

Now, we characterize the classes $z_j$, $\bar z_{i_0}$ and $z$ by the classes in $H^*(BSO(2n), BSO(2n-k))$ and the homomorphism $\bar f_n^*$.

THEOREM 2.7. *Let $0<k<2n$. The classes $z_j$, $\bar z_{i_0}$, $z$ and $z'$ in $H^*(Z_{2n,k})$ can be chosen so as to satisfy the following conditions* (2.7.1–5).

$$z' = \tilde f_n^* \chi' \qquad \textit{if } k \textit{ is even.} \tag{2.7.1}$$

*For the case* (*a*) *of Theorem* 1.11,

$$\delta z_j = \bar f_n^* U_j - \binom{n}{j} y^{2j-n} \bar f_n^* U \qquad (j = n-k'+1, \cdots, n-1). \tag{2.7.2}$$

*For the case* (*b*) *of Theorem* 1.11,

$$\begin{cases} \delta z_j = \bar f_n^* U_j - \binom{n}{j} y^{2j-n} \bar f_n^* U & (j = n-k'+1, \cdots, \hat{i}_0, \cdots, n-1), \\ \delta \bar z_{i_0} = \bar f_n^* U_{i_0} - \binom{n}{i_0} \bar f_n^* U. & \end{cases} \tag{2.7.3}$$

*For the case* (*c*) *of Theorem* 1.11,

$$\delta z_j = \bar f_n^* U_j + \lambda_j y^{2j-2i_0} \bar f_n^* U_{i_0} \qquad (j = n-k'+1, \cdots, \hat{i}_0, \cdots, n-1), \tag{2.7.4}$$

$$\delta z = \bar f_n^* U + \lambda_0 y^{n-2i_0} \bar f_n^* U_{i_0}, \tag{2.7.5}$$

*where $\lambda_j$ satisfies the formula* $\binom{n}{j} + \lambda_j \binom{n}{i_0} \equiv 0 \bmod p$.

*The generators of $H^*(X_{2n,k})$ are obtained by replacing $\tilde f_n^*$ with $\tilde\pi^* \tilde f_n^*$ and $\bar f_n^*$ with $\bar\pi^* \bar f_n^*$ in* (2.7.1–5).

PROOF. (2.7.1) has been proved in the proof of Theorem 1.11.

It is sufficient to prove (2.7.2–5) for odd $k$ by Lemma 2.1. By the diagram (2.6), we have

$$t^*\bar{f}_n^*U_j=f_n^*p_j=\binom{n}{j}y^{2j}, \qquad t^*\bar{f}_n^*U=f_n^*\chi=y^n. \tag{2.8}$$

Consider the case ($a$). Then there exists a unique class $z_j$ in $H^{4j-1}(Z_{2n,k})$ such that

$$\delta z_j=\bar{f}_n^*U_j-\binom{n}{j}y^{2j-n}\bar{f}_n^*U,$$

since the image of the right hand side by $t^*$ is zero by (2.8) and $\delta$ is monomorphic in odd degree. To see that the above classes $z_j$ ($j=n-k'+1, \dots, n-1$) are generators of (1.11.1), it is sufficient to show that $i^*z_j=v_j$ ($j=n-k'+1, \dots, n-1$). For this purpose we consider the following diagram induced by (2.4):

$$\begin{tikzcd}
H^{*-1}(V_{2n,k}) \arrow[r, "\delta_1", "\approx"'] & H^*(CV_{2n,k}, V_{2n,k}) & \\
& & H^*(BSO(2n), BSO(2n-k)) \arrow[ul, "g^*"'] \arrow[dl, "\bar{f}_n^*"] \\
H^{*-1}(Z_{2n,k}) \arrow[r, "\delta"] \arrow[uu, "i^*"] & H^*(CP^\infty, Z_{2n,k}) \arrow[uu, "h^*"] &
\end{tikzcd}$$

By Lemma 2.5, we have $\delta_1 v_j=g^*U_j$ and so we have

$$\delta_1 v_j=h^*\bar{f}_n^*U_j=h^*(\delta z_j+\binom{n}{j}y^{2j-n}\bar{f}_n^*U)=\delta_1 i^*z_j+\binom{n}{j}h^*y^{2j-n}h^*\bar{f}_n^*U=\delta_1 i^*z_j,$$

since $h^*y^{2j-n}=0$ in $H^*(CV_{2n,k})$. Therefore we obtain $i^*z_j=v_j$ because $\delta_1$ is isomorphic.

In the similar way, we can prove the theorem for the other cases.

Q.E.D.

## § 3. Reduced power operations $\mathcal{P}^i$ in $H^*(X_{2n,k})$ and $H^*(Z_{2n,k})$

In this section, we determine the mod $p$ reduced power operations $\mathcal{P}^i$ in $H^*(X_{2n,k})$ and $H^*(Z_{2n,k})$ for $n=p^r$ or $2p^r$, and also we notice that they are computable for any positive integers $n$ and $k$ ($0<k<2n$).

A. Borel and J.-P. Serre [4, § 14] studied the mod $p$ reduced power operations $\mathcal{P}^i$ in $H^*(BSO(2n))$:

$$\mathcal{P}^i p_j=(-1)^{qi}b_p^{i,2j+2qi}p_{j+qi}+\sum_{l=j}^{j+qi-1}p_l\alpha_l, \qquad \alpha_l\in\tilde{H}^*(BSO(2n)), \tag{3.1}$$

$$\mathcal{P}^i\chi=\chi C^{i,q}(p_1, \dots, p_{n-1}, \chi^2) \qquad (2q=p-1) \tag{3.2}$$

where $b_p^{i,2j+2qi}$ is an integer and $C^{i,q}(\dots)$ is given as follows: Let $\sigma_i$ be the $i$-th elementary symmetric function with respect to indeterminates $x_1, \dots, x_n$, then $C^{i,q}(\sigma_1, \cdots, \sigma_n)$ denotes the polynomial which expresses the symmetric polynomial of typical term $x_1^q \cdots x_i^q$.

Moreover, by S. Mukohda and S. Sawaki [10], it is known that

$$b_p^{i,2j+2qi} \equiv \binom{2j-1}{i} \bmod p. \tag{3.3}$$

First of all, we calculate $\mathcal{P}^i z'$ in $H^*(Z_{2n,k})$ and $H^*(X_{2n,k})$ when $k=2k'$. Since $z'=\tilde{f}_n^* \chi'$ by Theorem 2.7, we have

$$\begin{aligned}\mathcal{P}^i z' &= \mathcal{P}^i \tilde{f}_n^* \chi' = \tilde{f}_n^* \mathcal{P}^i \chi' = \tilde{f}_n^* (\chi' C^{i,q}(p_1, \cdots, p_{n-k'-1}, \chi'^2)) \\ &= z' C^{i,q}(\tilde{f}_n^* p_1, \cdots, \tilde{f}_n^* p_{n-k'-1}, \tilde{f}_n^* \chi'^2)\end{aligned}$$

and hence, we obtain

$$\mathcal{P}^i z' = z' C^{i,q}\left(\binom{n}{1} y^2, \cdots, \binom{n}{n-k'} y^{2n-k}\right). \tag{3.4}$$

Therefore we can calculate $\mathcal{P}^i z'$ for any $n$ and even $k$.

Now let $k=2k'-1$ and consider the following diagram of the exact sequences (cf. (2.6)):

$$\begin{array}{ccccccc} 0 \longrightarrow H^*(BSO(2n), BSO(2n-k)) & \xrightarrow{j^*} & H^*(BSO(2n)) & \xrightarrow{\pi'^*} & H^*(BSO(2n-k)) \longrightarrow 0 \\ \downarrow \bar{f}_n^* & & \downarrow f_n^* & & \downarrow \tilde{f}_n^* \\ \cdots \longrightarrow H^{*-1}(Z_{2n,k}) \xrightarrow{\delta} H^*(CP^\infty, Z_{2n,k}) & \xrightarrow{t^*} & H^*(CP^\infty) & \xrightarrow{p^*} & H^*(Z_{2n,k}) \longrightarrow \cdots. \end{array} \tag{3.5}$$

Then using (3.1–3), we have

$$\mathcal{P}^i U_j = \begin{cases} (-1)^{qi}\binom{2j-1}{i} U_{j+qi} + \sum_{l=j}^{j+qi-1} U_l \alpha_l & \text{for } j+qi \neq n \\ (-1)^{qi}\binom{2j-1}{i} U\chi + \sum_{l=j}^{j+qi-1} U_l \alpha_l & \text{for } j+qi = n, \end{cases} \tag{3.6}$$

$$\mathcal{P}^i U = U C^{i,q}(p_1, \cdots, p_{n-1}, \chi^2) \tag{3.7}$$

in $H^*(BSO(2n), BSO(2n-k))$, where $U_j$ and $U$ are the elements in (2.3):

$$j^* U_j = p_j \ (j=n-k'+1, \cdots, n-1), \qquad j^* U = \chi.$$

Mapping (3.6–7) by $\bar{f}_n^*$, we have

$$\bar{f}_n^* \mathcal{P}^i U_j = \begin{cases} (-1)^{qi}\binom{2j-1}{i} \bar{f}_n^* U_{j+qi} + \sum_{l=j}^{j+qi-1} \bar{f}_n^* U_l f_n^* \alpha_l & \text{for } j+qi \neq n \\ (-1)^{qi}\binom{2j-1}{i} \bar{f}_n^* U f_n^* \chi + \sum_{l=j}^{j+qi-1} \bar{f}_n^* U_l f_n^* \alpha_l & \text{for } j+qi = n, \end{cases} \tag{3.8}$$

$$\bar{f}_n^* \mathcal{P}^i U = \bar{f}_n^* U C^{i,q}\left(\binom{n}{1} y^2, \cdots, \binom{n}{n-1} y^{2n-2}, y^{2n}\right). \tag{3.9}$$

Using (3.4), (3.8–9) and Theorem 2.7, we have the following theorem.

THEOREM 3.10. *Let $n=p^r$ or $2p^r (r\geq 1)$ and $k$ be a positive integer such that $k<2n$. Then the* mod $p$ *reduced power operations $\mathcal{P}^i$ in $H^*(X_{2n,k})$ and $H^*(Z_{2n,k})$ are given by*

$$\mathcal{P}^i z_j = \begin{cases} (-1)^{qi}\binom{2j-1}{i} z_{j+qi} & \textit{for} \quad j<n-qi \\ 0 & \textit{for} \quad j\geq n-qi, \end{cases} \tag{3.10.1}$$

$$\mathcal{P}^i z' = 0 \qquad \textit{for} \quad i>0, \tag{3.10.2}$$

*where* $2q=p-1$.

PROOF. Assume that $k=2k'$, then we have

$$\mathcal{P}^i z' = \begin{cases} z'C^{i,q}(0, \dots, 0) & \text{if} \quad n=p^r \text{ or } 2p^r;\ k'>p^r \\ z'C^{i,q}(0, \dots, 0, 2y^{2p^r}, 0, \dots, 0) & \text{if} \quad n=2p^r \text{ and } k'\leq p^r, \end{cases}$$

by (3.4). According to Theorem 1.11 and Remark 1.12, we have $y^{2p^r}=0$ in $H^*(X_{2n,k})$ and $H^*(Z_{2n,k})$, and so we obtain (3.10.2).

We shall prove (3.10.1) for odd $k$. Then (3.10.1) for even $k$ follows from Lemma 2.1.

Now $\alpha_l \in \tilde{H}^*(BSO(2n))$ is a polynomial of $p_j (j=1, \dots, n)$ and $f_n^* p_j = \binom{n}{j} y^{2j}$. Therefore $f_n^* \alpha_l$ has a common factor $y^{2p^r}$ and so we notice that

$$p^* f_n^* \alpha_l = 0 \qquad \text{for} \quad \alpha_l \in \tilde{H}^*(BSO(2n)),$$

since $p^* y^{2p^r}=0$ in $H^*(Z_{2n,k})$.

By (2.7.2–3) and Remark 1.12, we have

$$\delta z_j = \bar{f}_n^* U_j - a\bar{f}_n^* U, \qquad a= \begin{cases} 2 & \text{if } n=2p^r \text{ and } j=p^r \\ 0 & \text{otherwise.} \end{cases}$$

Using (3.8–9) and the above facts, we have

$$\delta \mathcal{P}^i z_j = \mathcal{P}^i(\bar{f}_n^* U_j - a\bar{f}_n^* U)$$

$$= \begin{cases} (-1)^{qi}\binom{2j-1}{i} \bar{f}_n^* U_{j+qi} + \sum_{l=j}^{j+qi-1} \bar{f}_n^* U_l f_n^* \alpha_l - a\bar{f}_n^* U C^{i,q}\left(\binom{n}{1} y^2, \dots, \binom{n}{n} y^{2n}\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad j+qi \neq n \\ (-1)^{qi}\binom{2j-1}{i} \bar{f}_n^* U f_n^* \chi + \sum_{l=j}^{j+qi-1} \bar{f}_n^* U_l f_n^* \alpha_l - a\bar{f}_n^* U C^{i,q}\left(\binom{n}{1} y^2, \dots, \binom{n}{n} y^{2n}\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad j+qi = n \end{cases}$$

$$= \begin{cases} (-1)^{qi}\binom{2j-1}{i}\delta z_{j+qi} + \delta\left\{\sum_{l=j}^{j+qi-1}(p^*f_n^*\alpha_l)z_l\right\} + A\bar{f}_n^* U & \text{if } \ j+qi \neq n \\ \delta\left\{\sum_{l=j}^{j+qi-1}(p^*f_n^*\alpha_l)z_l\right\} + A'\bar{f}_n^* U & \text{if } \ j+qi = n \end{cases}$$

$$= \begin{cases} (-1)^{qi}\binom{2j-1}{i}\delta z_{j+qi} + A\bar{f}_n^* U & \text{if } \ j+qi \neq n \\ A'\bar{f}_n^* U & \text{if } \ j+qi = n, \end{cases}$$

where $A$, $A' \in H^*(CP^\infty)$. Mapping this equality by $t^*$ and using (2.8), we have $A=0$ and $A'=0$. Since $\delta$ is monomorphic in odd degree, (3.10.1) follows.

Q.E.D.

THEOREM 3.11. *Let $n$ and $k$ be positive integers with $k<2n$, satisfying $n=n'p^r$, $r\geq 1$, $n'\geq 3$, $(p, n')=1$ and $n-[(k+1)/2]+1\leq p^r$. Then the cohomology algebras of $X_{2n,k}$ and $Z_{2n,k}$ are the case (c) of Theorem 1.11 with $N(n, k)=4p^r<2n$ and the* mod $p$ *reduced power operations $\mathcal{P}^i (i>0)$ in $H^*(X_{2n,k})$ and $H^*(Z_{2n,k})$ are given by*

$$\mathcal{P}^i z_j = \begin{cases} (-1)^{qi}\binom{2j-1}{i} z_{j+qi} & \text{for } \ j<n-qi,\ j\neq p^r-qi \\ 0 & \text{otherwise,} \end{cases} \tag{3.11.1}$$

$$\mathcal{P}^i z = \begin{cases} \binom{n}{i} y^{2qi} z - \frac{(-1)^{qi}}{n'}\binom{2p^r-1}{i} y^{n-2p^r} z_{p^r+qi} & \text{for } \ p^r+qi<n \\ \binom{n}{i} y^{2qi} z & \text{for } \ p^r+qi\geq n, \end{cases} \tag{3.11.2}$$

$$\mathcal{P}^i z' = 0, \tag{3.11.3}$$

*where $2q=p-1$.*

PROOF. It is clear that $N(n, k)=4p^r=4i_0<2n$ by the assumptions and so $y^{2p^r}=0$ in $H^*(Z_{2n,k})$. Hence we have

$$p^*f_n^*\alpha_l = 0 \qquad \text{for} \quad \alpha_l \in \tilde{H}^*(BSO(2n)),$$

similarly to the proof of Theorem 3.10.

We notice that the integers $\lambda_j$ of (2.7.4) are zero if $j\neq lp^r$ ($l=2, \dots, n'-1$), and $\lambda_0$ of (2.7.5) is equal to $-1/n'$. Therefore, using (2.7.4–5), (3.8–9) and $y^{2p^r}=0$, we have

$$\delta\mathcal{P}^i z_j = \mathcal{P}^i(\bar{f}_n^* U_j + \lambda_j y^{2j-2i_0}\bar{f}_n^* U_{i_0})$$

$$= \begin{cases} (-1)^{qi}\binom{2j-1}{i}\delta z_{j+qi} + A\bar{f}_n^* U_{i_0} & \text{for } \ j+qi<n,\ j+qi\neq i_0 \\ A'\bar{f}_n^* U_{i_0} & \text{otherwise,} \end{cases}$$

$$\delta\mathcal{P}^i z = \mathcal{P}^i(\bar{f}_n^* U + \lambda_0 y^{n-2i_0} \bar{f}_n^* U_{i_0})$$

$$= \begin{cases} \binom{n}{i} y^{2qi} \delta z - \dfrac{(-1)^{qi}}{n'} \binom{2p^r - 1}{i} y^{n-2p^r} \delta z_{p^r+qi} + \bar{A} \bar{f}_n^* U_{i_0} & \text{for} \quad i_0 + qi < n \\ \binom{n}{i} y^{2qi} \delta z + \bar{A}' \bar{f}_n^* U_{i_0} & \text{for} \quad i_0 + qi \geq n. \end{cases}$$

Here $A$, $A'$, $\bar{A}$ and $\bar{A}'$ are some elements of $H^*(CP^\infty)$, and we see that these elements are zero in the same way as the proof of (3.10.1). Hence (3.11.1) and (3.11.2) follow.

(3.11.3) is obtained similarly to (3.10.2). Q.E.D.

In general, the mod $p$ reduced power operations $\mathcal{P}^i$ in $H^*(X_{2n,k})$ and $H^*(Z_{2n,k})$ are given by the following

Proposition 3.12. *For any positive integers $n$ and $k$ ($k < 2n$), the* mod $p$ *reduced power operations $\mathcal{P}^i$ in $H^*(X_{2n,k})$ and $H^*(Z_{2n,k})$ are given as follows:*

(3.12.1) $$\mathcal{P}^i z_j = (-1)^{qi} \binom{2j-1}{i} z_{j+qi} + \sum_l a_l y^{2j+2qi-2l} z_l,$$

(3.12.2) $$\mathcal{P}^i z = \binom{n}{i} y^{2qi} z + \sum_{l=i_0+1}^{i_0+qi} a'_l y^{n+2qi-2l} z_l,$$

(3.12.3) $$\mathcal{P}^i z' = z' C^{i,q}\left(\binom{n}{1} y^2, \cdots, \binom{n}{n-k'} y^{2n-k}\right) \qquad (k = 2k'),$$

*where $\sum_l$ in* (3.12.1) *is the sum of $l = j, \cdots, j+qi-1$ for the case* (*a*), (*b*) *and $l = \min\{j, i_0+1\}, \cdots, j+qi-1$ for the case* (*c*) *of Theorem* 1.11, *and $a_l$, $a'_l$ are some integers.*

Proof. We have already proved (3.12.3) in (3.4).

For the case (*c*) of Theorem 1.11, we have

$$\delta\mathcal{P}^i z_j = \mathcal{P}^i \delta z_j = \mathcal{P}^i(\bar{f}_n^* U_j + \lambda_j y^{2j-2i_0} \bar{f}_n^* U_{i_0})$$

$$= \begin{cases} (-1)^{qi} \binom{2j-1}{i} \delta z_{j+qi} + \sum_l a_l y^{2j+2qi-2l} \delta z_l + A \bar{f}_n^* U_{i_0} & \text{for} \quad j \neq n - qi \\ (-1)^{qi} \binom{2j-1}{i} a' y^n \delta z + \sum_l \bar{a}_l y^{2j+2qi-2l} \delta z_l + \bar{A} \bar{f}_n^* U_{i_0} & \text{for} \quad j = n - qi, \end{cases}$$

$$\delta\mathcal{P}^i z = \mathcal{P}^i \delta z = \mathcal{P}^i(\bar{f}_n^* U + \lambda_0 y^{n-2i_0} \bar{f}_n^* U_{i_0})$$

$$= \binom{n}{i} y^{2qi} \delta z + \sum_{l=i_0+1}^{i_0+qi} a'_l y^{n+2qi-2l} \delta z_l + A' \bar{f}_n^* U_{i_0},$$

by (2.7.4–5) and (3.8–9), where $a_l$, $\bar{a}_l$ and $a'_l$ are some integers and $A$, $\bar{A}$ and $A'$ are some elements in $H^*(CP^\infty)$. In the similar way to the proof of Theorem 3.10, we have (3.12.1) and (3.12.2).

For the case ($a$) or ($b$) of Theorem 1.11, we have (3.12.1) similarly.

Q.E.D.

## § 4. Bockstein homomorphisms $\beta$ in $H^*(X_{2n,k})$ and $H^*(Z_{2n,k})$

P. F. Baum and W. Browder [1] determined the mod $p$ cohomology algebra of the projective unitary group $PU(n)=U(n)/S^1$ and the reduced power operations $\mathcal{P}^i$ when $n=n'p^r$, $(p, n')=1$ and $r\geq 1$. Moreover, they determined the Bockstein homomorphism $\beta$ in degree $\leq 2p^{r-1}$. According to [1, MAIN THEOREM I], the mod $p$ cohomology structure of $PU(n)$ is the following:

Let $n=n'p^r$, $(p, n')=1$ and $r\geq 1$. Then

$$H^*(PU(n))=\wedge(w_1, \cdots, \hat{w}_{p^r}, \cdots, w_n)\otimes Z_p[y]/(y^{p^r})$$

where deg $w_j=2j-1$ and deg $y=2$,

$$\beta w_j=\begin{cases}\mu y^{p^{r-1}}, & \mu\not\equiv 0 \bmod p, \quad \text{for} \quad j=p^{r-1}\\ 0 & \qquad\qquad\qquad\quad \text{for} \quad j<p^{r-1}.\end{cases} \tag{4.1}$$

REMARK. It is proved that $\beta w_j=0$ for $j<p^{r-1}$ of (4.1), in the proof of MAIN THEOREM I in [1, p. 324].

First, we shall extend (4.1) for all $j$ $(1\leq j\leq n, j\neq p^r)$. For this purpose, we use the properties of generators $w_j$ in $H^*(PU(n))$.

Let $EU(n)$ be a contractible space such that $U(n)$ acts freely, then $EU(n)/U(n)=BU(n)$ is a classifying space of $U(n)$, and there is the following homotopy commutative diagram ([12, §§1–2]):

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d, equal] & U(n) \arrow[r, "i"] \arrow[d] & PU(n) \arrow[r, phantom, "\simeq"] \arrow[d] & S^\infty\times_{S^1}U(n) \arrow[d, "p"'] \\ S^1 \arrow[r] & EU(n) \arrow[r] \arrow[d, "\pi'"'] & EU(n)/S^1 \arrow[r, phantom, "\simeq"] \arrow[d] & CP^\infty \arrow[dl, "f_n"] \\ & BU(n) \arrow[r, equal] & BU(n) & \end{tikzcd} \tag{4.2}$$

where $f_n$ is a classifying map of $n\xi$. Then we obtain the following diagram induced by (4.2):

$$\begin{tikzcd} 0 \arrow[r] & H^{2j-1}(PU(n)) \arrow[r, "\delta"] \arrow[d, "i^*"'] & H^{2j}(CP^\infty, PU(n)) \arrow[r, "i^*"] \arrow[d, "\bar{i}^*"'] & H^{2j}(CP^\infty) \arrow[r] & \cdots \\ & H^{2j-1}(U(n)) \arrow[r, "\delta", "\approx"'] & H^{2j}(EU(n), U(n)) & H^{2j}(BU(n), *) \arrow[l, "\pi'^*"'] \arrow[ul, "f_n^*"'] \arrow[r, equal] & H^{2j}(BU(n)). \arrow[ul, "f_n^*"'] \end{tikzcd} \tag{4.3}$$

The cohomology algebras of $BU(n)$ and $U(n)$ are given as follows:

$$H^*(BU(n))=Z_p[c_1, \cdots, c_n], \qquad H^*(U(n))=\wedge(u_1, \cdots, u_n),$$

where the element $c_j$ is the universal $j$-th Chern class and the element $u_j$ is transgressive and $\tau u_j = c_j$.

Since the $j$-th Chern class of $n\xi$ over $CP^\infty$ is $\binom{n}{j}y^j$, we obtain $t^*\bar{f}_n^*c_j = f_n^*c_j = \binom{n}{j}y^j$ in (4.3).

LEMMA 4.4. *Let $n = n'p^r$, $(p, n') = 1$ and $r \geq 1$. We can choose the generators $w_j \in H^*(PU(n))$ $(j = 1, \cdots, \widehat{p^r}, \cdots, n)$ such that*

$$\delta w_j = \begin{cases} \bar{f}_n^*c_j - \dfrac{1}{n'^l}\dbinom{n'}{l}\bar{f}_n^*c_{p^r}^l & \text{if } j = lp^r, \quad l = 2, \cdots, n' \\ \bar{f}_n^*c_j & \text{otherwise.} \end{cases} \tag{4.4.1}$$

PROOF. If $p^r$ does not divide $j$, then $t^*\bar{f}_n^*c_j = 0$. Therefore we have a unique element $w_j \in H^{2j-1}(PU(n))$ such that $\delta w_j = \bar{f}_n^*c_j$. If $j = lp^r$, we have a unique element $w_j \in H^{2j-1}$ $(PU(n))$ such that $\delta w_j = \bar{f}_n^*c_j - \frac{1}{n'^l}\binom{n'}{l}\bar{f}_n^*c_{p^r}^l$. Using the diagram (4.3), we have $i^*w_j = u_j$. Therefore the lemma follows from the proof of [1, Corollary 4.2]. Q.E.D.

LEMMA 4.5. *Let $n = n'p^r$, $(p, n') = 1$ and $r \geq 1$. Then the Bockstein homomorphism $\beta$ in $H^*(PU(n))$ is given as follows:*

$$\beta w_j = \begin{cases} \mu_l y^j & \text{for } j = lp^{r-1} \quad (l = 1, \cdots, p-1) \\ 0 & \text{otherwise,} \end{cases}$$

*where $\mu_l = \frac{1}{l}\binom{p-1}{l-1}\mu$ and $\mu$ is the one of* (4.1).

PROOF. By Lemma 4.4, we have $\delta\beta w_j = 0$ and so $\beta w_j \in p^*H^*(CP^\infty)$. Therefore $\beta w_j = 0$ for $j > p^r$.

Assume that $p^{r-1} < j < p^r$. Now, we use the same notations in the integral cohomology $H^*(\ ; Z)$ of $BU(n)$ and $CP^\infty$. Set $k = p^{r-1}$ and consider the element $x_j = a\bar{f}_n^*c_j - a_j y^{j-k}\bar{f}_n^*c_k$ in $H^{2j}(CP^\infty, PU(n); Z)$, where $a = \frac{1}{p}\binom{n}{k} \equiv n'$ mod $p$ and $a_j = \frac{1}{p}\binom{n}{j}$. Since $t^*x_j = a\binom{n}{j}y^j - a_j\binom{n}{k}y^j = 0$ in $H^{2j}(CP^\infty; Z)$, there exists an element $x'_j \in H^{2j-1}(PU(n); Z)$ such that $\delta x'_j = x_j$. Therefore we have

$$\begin{aligned} \delta\rho_p x'_j &= \rho_p\delta x'_j = \rho_p(a\bar{f}_n^*c_j - a_j y^{j-k}\bar{f}_n^*c_k) \\ &= a\bar{f}_n^*c_j - a_j y^{j-k}\bar{f}_n^*c_k = \delta(aw_j - a_j y^{j-k}w_k) \end{aligned}$$

in $H^{2j}(CP^\infty, PU(n))$ by Lemma 4.4, where $\rho_p$ is the mod $p$ reduction. Since $\delta$

is monomorphic in degree $2j-1$, we obtain $\rho_p x'_j = aw_j - a_j y^{j-k} w_k$. Using (4.1) and the fact $\beta\rho_p = 0$, we obtain $0 = \beta\rho_p x'_j = a\beta w_j - a_j \mu y^j$. Therefore we have

$$\beta w_j = a^{-1} a_j \mu y^j.$$

By the simple calculations it is proved that

$$\frac{1}{p}\binom{n'p^r}{j} \equiv \begin{cases} \dfrac{n'}{l}\dbinom{p-1}{l-1} & \text{if } j = lp^{r-1}, \quad l = 1, \cdots, p-1 \\ 0 & \text{otherwise} \end{cases}$$

mod $p$. Therefore we have the lemma. Q.E.D.

Let $h: U(n) \longrightarrow SO(2n)$ be the natural inclusion. Then, we have the following homotopy commutative diagram of fibrations:

(4.6)
$$\begin{tikzcd} U(n) \arrow[r] \arrow[d, "h"'] & PU(n) \arrow[r] \arrow[d, "\tilde{h}"'] & CP^\infty \arrow[d, equal] \\ SO(2n) \arrow[r] & SO(2n)/S^1 = Z_{2n,2n-1} \arrow[r, "p"] & CP^\infty \end{tikzcd}$$

and the commutative diagram of the exact sequences induced by the map $h$:

(4.7)
$$\begin{tikzcd} \cdots \arrow[r] & H^{*-1}(Z_{2n,2n-1}) \arrow[r, "\delta"] \arrow[ddd, "\tilde{h}^*"'] & H^*(CP^\infty, Z_{2n,2n-1}) \arrow[rr, "t^*"] \arrow[ddd, "\tilde{h}^*"'] & & H^*(CP^\infty) \arrow[r] & \cdots \\ & & & H^*(BSO(2n), *) \arrow[ul, "\bar{f}_n^*"'] \arrow[rr, equal] \arrow[d, "\hat{h}^*"'] & & H^*(BSO(2n)) \arrow[ul, "f_n^*"'] \arrow[d, "\hat{h}^*"'] \\ & & & H^*(BU(n), *) \arrow[dl, "\bar{f}^*"] \arrow[rr, equal] & & H^*(BU(n)) \arrow[dl, "f_n^*"] \\ \cdots \arrow[r] & H^{*-1}(PU(n)) \arrow[r, "\delta"] & H^*(CP^\infty, PU(n)) \arrow[rr, "t^*"] & & H^*(CP^\infty) \arrow[r] & \cdots. \end{tikzcd}$$

The homomorphism $\hat{h}^*: H^*(BSO(2n)) \longrightarrow H^*(BU(n))$ is given as follows (e.g. [9]):

(4.8)
$$\hat{h}^* p_j = \sum_{k+l=2j} (-1)^{j+k} c_k c_l,$$

(4.9)
$$\hat{h}^* \chi = c_n.$$

LEMMA 4.10. *Let $n = n'p^r$, $(p, n') = 1$ and $r \geq 1$. Then the homomorphism $\tilde{h}^*: H^*(Z_{2n,2n-1}) \longrightarrow H^*(PU(n))$ is given by*

(4.10.1)
$$\tilde{h}^* y = y,$$

(4.10.2)
$$\tilde{h}^* z_j = \begin{cases} 0 & \text{for } j > n/2 \\ -4w_n & \text{for } j = n/2, \ n = 2p^r \\ (-1)^j 2w_{2j} & \text{otherwise,} \end{cases}$$

(4.10.3)
$$\tilde{h}^* z = w_n.$$

*Moreover, $\bar{h}^*$ is a monomorphism in degree smaller than $2p^r$.*

PROOF. Assume that $n'\geq 3$. Then $N(n, 2n-1)=4p^r=4i_0$ and $H^*(Z_{2n,2n-1})$ is the case $(c)$ of Theorem 1.11. Furthermore, in the equality (2.7.4):

$$\delta z_j=\bar{f}_n^* U_j+\lambda_j y^{2j-2i_0}\bar{f}_n^* U_{i_0} \qquad (j=1, \cdots, \hat{i}_0, \cdots, n-1),$$

we have $\lambda_j\equiv 0 \bmod p$ if $j\neq lp^r$ $(l=2, \cdots, n'-1)$. On the other hand,

$$\bar{f}_n^*(c_{2j-s}c_s)=\bar{f}_n^* c_{2j-s}f_n^* c_s=\binom{n}{s}\delta w_{2j-s}y^s+A_{j,s}\bar{f}_n^* c_{i_0},$$

where $A_{j,s}\in H^*(CP^\infty)$, by (4.4.1). In this equality, $\binom{n}{s}\equiv 0 \bmod p$ if $s\neq lp^r$ $(l>0)$ and $\delta w_{2j-s}y^s=\delta(w_{2j-s}y^s)=0$ if $s=lp^r$ $(l>0)$. By these facts and (4.7–8), we have

$$\begin{aligned}\delta\bar{h}^* z_j&=\bar{h}^*\delta z_j=\bar{f}_n^*\hat{h}^* U_j+\lambda_j y^{2j-2i_0}\bar{f}_n^*\hat{h}^* U_{i_0}\\ &=\bar{f}_n^*\Big(\sum_{s=0}^{2j}(-1)^{j+s}c_{2j-s}c_s\Big)-\lambda_j y^{2j-2i_0}\bar{f}_n^*\Big(\sum_{t=0}^{2i_0}(-1)^t c_{2i_0-t}c_t\Big)\\ &=\begin{cases}(-1)^j 2\delta w_{2j}+A\bar{f}_n^* c_{i_0} & \text{if } j\leq n/2\\ A'\bar{f}_n^* c_{i_0} & \text{if } j>n/2,\end{cases}\end{aligned}$$

where $A, A'\in H^*(CP^\infty)$. Mapping this equality by $t^*$ and using the fact $t^*\bar{f}_n^* c_{i_0}=n'y^{i_0}\neq 0$, we have $A=0$ and $A'=0$. Since $\delta$ is a monomorphism in odd degree, we have (4.10.2) for $n'\geq 3$.

For the case $n'=2$, $N(n, 2n-1)=4p^r=2n$ and $H^*(Z_{2n,2n-1})$ is the case $(b)$ of Theorem 1.11. Therefore

$$\begin{aligned}\delta\bar{h}^* z_j&=\bar{h}^*(\bar{f}_n^* U_j-\binom{n}{j}y^{2j-n}\bar{f}_n^* U)\\ &=\begin{cases}(-1)^j 2\delta w_{2j}-\binom{n}{j}y^{2j-n}\delta w_n & \text{if } j\leq n/2\\ 0 & \text{if } j>n/2,\end{cases}\end{aligned}$$

by (2.7.3) and (4.7–9), and so we have (4.10.2) for $n'=2$, similarly. (4.10.2) for $n'=1$ and (4.10.3) are proved in the same way. Q.E.D.

There exists a fibration $V_{2n-k+2,2}\longrightarrow V_{2n,k}\xrightarrow{\nu_k} V_{2n,k-2}$, where $\nu_k$ is the natural projection. This fibration induces fibrations $V_{2n-k+2,2}\longrightarrow Z_{2n,k}\xrightarrow{\nu_k} Z_{2n,k-2}$ and $V_{2n-k+2,2}\longrightarrow X_{2n,k}\xrightarrow{\nu_k} X_{2n,k-2}$. If $k=2k'-1$, $\nu_k^*$: $H^*(V_{2n,k-2})=\wedge(v_{n-k'+2}, \cdots, v_{n-1}, v)\longrightarrow H^*(V_{2n,k})=\wedge(v_{n-k'+1}, \cdots, v_{n-1}, v)$ is given as follows ([2, §10]):

$$\nu_k^* v_j = v_j, \qquad \nu_k^* v = v.$$

And so we have the following lemma.

LEMMA 4.11. *Let $k=2k'-1$. If $N(n, k)=4i_0$ exists, then assume that $i_0 \geq n/2$ or $i_0 \neq n-k'+1$. Then the homomorphisms $\nu_k^*$: $H^*(Z_{2n,k-2}) \longrightarrow H^*(Z_{2n,k})$, $H^*(X_{2n,k-2}) \longrightarrow H^*(X_{2n,k})$ are given as follows:*

$$\nu_k^* z_j = z_j \qquad \text{for} \quad n-k'+2 \leq j \leq n-1,$$

$$\nu_k^* z = z, \qquad \nu_k^* x = x, \qquad \nu_k^* y = y.$$

*Moreover $\nu_k^*$ are monomorphic.*

By the induction on $k$, we determine the Bockstein homomorphism $\beta$ in $H^*(Z_{2n,k})$ and $H^*(X_{2n,k})$ when $n$, $k$ satisfy $(*)$ of below.

THEOREM 4.12. *Let $n$ and $k$ be positive integers with $k<2n$, satisfying*

$$(*) \qquad n=n'p^r,\ r\geq 1,\ (p, n')=1;\ n-[(k+1)/2]+1 \leq p^r \text{ if } n' \geq 3.$$

*Then the Bockstein homomorphisms in $H^*(X_{2n,k})$ and $H^*(Z_{2n,k})$ are given by*

$$(4.12.1) \qquad \beta z_j = \begin{cases} \mu_l y^{2j} & \text{for} \quad j<n/2,\ j=lp^{r-1}\ (l=1, \cdots, p-1) \\ 0 & \text{otherwise,} \end{cases}$$

$$(4.12.2) \qquad \beta z=0, \qquad \beta z'=0, \qquad \beta x=y, \qquad \beta y=0,$$

*where $\mu_l$ is the same as in Lemma 4.5.*

PROOF. The last two relations of (4.12.2) follow from (1.10). It follows easily that $\beta z'=0$ by the dimensional reason. According to Theorem 2.7, we have $\delta\beta z_j=0$ and $\delta\beta z=0$ and so $\beta z_j$ and $\beta z$ are the elements of $p^*H^*(CP^\infty)$. Therefore $\beta z_j=0$ for $j\geq p^r$ and $\beta z=0$, since $y^{2p^r}=0$ in $H^*(Z_{2n,k})$ under the assumption $(*)$ (cf. the proof of Theorems 3.10 and 3.11).

By Lemmas 2.1 and 4.11, it is sufficient to prove (4.12.1) in $H^*(Z_{2n,2n-1})$ for $j<p^r$. By Lemmas 4.5 and 4.10, we have

$$\tilde{h}^*\beta z_j = (-1)^j 2\beta w_{2j} = \begin{cases} -2\mu_2 y^{2p^{r-1}} & \text{for} \quad j=p^{r-1} \\ 0 & \text{for} \quad j<p^{r-1} \end{cases}$$

$$= \begin{cases} -2\mu_2 \tilde{h}^* y^{2p^{r-1}} & \text{for} \quad j=p^{r-1} \\ 0 & \text{for} \quad j<p^{r-1}. \end{cases}$$

Since $-2\mu_2 = -(p-1)\mu \equiv \mu \bmod p$, $\tilde{h}^* y = y$ and $\tilde{h}^*$ is monomorphic in degree smaller than $2p^r$, it follows that

(4.13)
$$\beta z_j = \begin{cases} \mu y^{2p^{r-1}} & \text{for } j = p^{r-1} \\ 0 & \text{for } j < p^{r-1}. \end{cases}$$

Replace $c_j \in H^*(BU(n); Z)$ and $y \in H^*(CP^\infty; Z)$ in the proof of Lemma 4.5 with $U_j \in H^*(BSO(2n), *; Z)$ and $y^2 \in H^*(CP^\infty; Z)$, then we obtain (4.12.1) for $j < p^r$ by the entirely same technique as the proof of Lemma 4.5, using (4.13) and (2.7.2–4) in place of (4.1) and (4.4.1). Q.E.D.

REMARK. If $n = p^r$ in Theorem 4.12, then (4.12.1) is shown by Lemmas 4.5 and 4.10 only.

## § 5. The relations between $X_{2n,k}$ and $X_{2n+2m,k}$ and between $Z_{2n,k}$ and $Z_{2n+2m,k}$

We consider the following homotopy commutative diagram:

(5.1)
$$\begin{tikzcd} Z_{2n,k} \arrow[r, "d"] \arrow[d, dashed, "\rho"'] \arrow[dr, "p"] & Z_{2n,k} \times Z_{2n,k} \arrow[r, "1\times p"] & Z_{2n,k} \times CP^\infty \arrow[r] \arrow[d, "p\times 1"] & {} \\ Z_{2n+2m,k} \arrow[r, "p'"] & CP^\infty \arrow[r, "d"] & CP^\infty \times CP^\infty \arrow[r] & {} \end{tikzcd}$$
$$\begin{tikzcd} {} \arrow[r, "\tilde f_n \times f_m"] & BSO(2n-k) \times BSO(2m) \arrow[r, "\mu'"] \arrow[d, "\pi' \times 1"] & BSO(2n+2m-k) \arrow[d, "\pi''"] \\ {} \arrow[r, "f_n \times f_m"] & BSO(2n) \times BSO(2m) \arrow[r, "\mu"] & BSO(2n+2m). \end{tikzcd}$$

Here $d$ is the diagonal map, $\mu$ and $\mu'$ are the multiplications, $f_n$ and $f_m$ are classifying maps of $n\xi$ and $m\xi$, respectively. Then $(2n+2m)$-plane bundle $(n+m)p^*\xi$ has a map $\mu(f_n \times f_m)dp$ as a classifying map and $\mu(f_n \times f_m)dp$ is lifted to $\mu'(\tilde f_n \times f_m)d' : Z_{2n,k} \longrightarrow BSO(2n+2m-k)$, where $d' = (1 \times p)d : Z_{2n,k} \longrightarrow Z_{2n,k} \times CP^\infty$. Therefore the associated $V_{2n+2m,k}$-bundle of $(n+m)p^*\xi$ over $Z_{2n,k}$ has a cross section and so we obtain a map $\rho : Z_{2n,k} \longrightarrow Z_{2n+2m,k}$ such that $\rho^* p'^* \xi = p^* \xi$. Similarly, we have a map $\rho : X_{2n,k} \longrightarrow X_{2n+2m,k}$.

In this section, we use the same notations for the generators of $H^*(Z_{2n,k})$ (resp. $H^*(X_{2n,k})$) and $H^*(Z_{2n+2m,k})$ (resp. $H^*(X_{2n+2m,k})$).

THEOREM 5.2. *Let* $0 < k < 2n$ *and set* $N = N(n+m, k) = 4i_0$, $N' = N(n, k) = 4i_0'$, *and* $K(j) = \{s \mid j-n+1 \leq s \leq m\}$, $K'(j) = \{s \mid j-n+1 \leq s \leq m,\ s \neq j - i_0'\}$. *Then the homomorphisms* $\rho^* : H^*(Z_{2n+2m,k}) \longrightarrow H^*(Z_{2n,k})$ *and* $\rho^* : H^*(X_{2n+2m,k}) \longrightarrow H^*(X_{2n,k})$ *are given as follows:*

(5.2.1)
$$\rho^* x = x. \qquad \rho^* y = y,$$

(a) *If* $N$ *exists and* $2n+2m > N = 4i_0$, *then*

(5.2.2) $\rho^* z_j = \sum_{s\in K'(j)} \binom{m}{s} y^{2s} z_{j-s} + \lambda_j \sum_{t\in K'(i_0)} \binom{m}{t} y^{2j-2i_0+2t} z_{i_0-t}$ *when $N'$ exists and $2n > N'$,*

(5.2.3) $\rho^* z_j = \sum_{s\in K(j)} \binom{m}{s} y^{2s} z_{j-s} + \lambda_j \sum_{t\in K(i_0)} \binom{m}{t} y^{2j-2i_0+2t} z_{i_0-t}$ *otherwise,*

$$(5.2.4)\qquad \rho^* z = \begin{cases} y^m z + \lambda_0 \sum_{t\in K'(i_0)} \binom{m}{t} y^{n+m-2i_0+2t} z_{i_0-t} & \text{when } N' \text{ exists and } 2n > N' \\ \lambda_0 \sum_{t\in K(i_0)} \binom{m}{t} y^{n+m-2i_0+2t} z_{i_0-t} & \text{otherwise,} \end{cases}$$

*where $\lambda_j$ satisfies the formula* $\binom{n+m}{j} + \lambda_j \binom{n+m}{i_0} \equiv 0 \bmod p$.

(*b*) *If $N$ exists and $2n+2m \leq N$ or $N$ does not exist, then*

(5.2.5) $\rho^* z_j = \sum_{s\in K'(j)} \binom{m}{s} y^{2s} z_{j-s} - \binom{n+m}{j} y^{2j-n} z$ *when $N'$ exists and $2n > N'$,*

(5.2.6) $\rho^* z_j = \sum_{s\in K(j)} \binom{m}{s} y^{2s} z_{j-s}$ *otherwise.*

PROOF. (5.2.1) follows from $p'\rho \simeq p$. From the diagram (5.1) and the mapping-cylinder considerations, we have the following commutative diagram:

(5.3)

$$\begin{array}{ccccc} & H^{*-1}(Z_{2n+2m,k}) \xrightarrow{\delta} & H^*(CP^\infty, Z_{2n+2m,k}) & \xleftarrow{\bar f^*_{n+m}} & H^*(BSO(2n+2m), BSO(2n+2m-k)) \\ {}^{p'^*}\nearrow & \downarrow \rho^* & \downarrow \bar\rho^* & & \downarrow \bar\mu^* \\ H^{*-1}(CP^\infty) & & & & H^*((BSO(2n), BSO(2n-k)) \times BSO(2m)) \\ {}_{p^*}\searrow & \downarrow & \downarrow & & \downarrow (\bar f_n \times f_m)^* \\ & H^{*-1}(Z_{2n,k}) \xrightarrow{\delta} & H^*(CP^\infty, Z_{2n,k}) & \xleftarrow{\bar d^*} & H^*((CP^\infty, Z_{2n,k}) \times CP^\infty). \end{array}$$

It is well-known that

$$\mu^* p_j = \sum_{s+t=j} p'_s \times p''_t, \qquad \mu^* \chi = \chi' \times \chi'',$$

where $p_j$, $p'_j$, $p''_j$ and $\chi$, $\chi'$, $\chi''$ are the $j$-th Pontrjagin classes and the Euler classes of the universal oriented $(2n+2m)$-, $2n$-, $2m$-plane bundles. Therefore we obtain

(5.4) $\bar\mu^* U_j = \sum_{s\in K(j)} U'_{j-s} \times p''_s + U'^2 \times p''_{j-n} = \sum_{s\in K(j)} U'_{j-s} \times p_s + U'\chi' \times p''_{j-n}$,

(5.5) $\bar\mu^* U = U' \times \chi''$,

where $U$, $U_j$, $U'$ and $U'_j$ are the elements determined by (2.2–3).

Consider the case (*a*). Using (2.7.4) for $n+m$ and (5.3–5), we have

$$\begin{aligned}\delta\rho^* z_j &= \bar{\rho}^*\delta z_j = \bar{\rho}^*(\bar{f}^*_{n+m}U_j + \lambda_j y^{2j-2i_0}\bar{f}^*_{n+m}U_{i_0}) \\ &= \bar{d}^*(\bar{f}_n\times f_m)^*\bar{\mu}^*U_j + \lambda_j y^{2j-2i_0}\bar{d}^*(\bar{f}_n\times f_m)^*\bar{\mu}^*U_{i_0} \\ &= \sum_{s\in K(j)} \bar{f}^*_n U'_{j-s}p_s(m\xi) + \bar{f}^*_n U' y^n p_{j-n}(m\xi) \\ &\quad + \lambda_j y^{2j-2i_0}\Big\{\sum_{t\in K(i_0)} \bar{f}^*_n U'_{i_0-t}p_t(m\xi) + \bar{f}^*_n U' y^n p_{i_0-n}(m\xi)\Big\}.\end{aligned}$$

Assume that $N'$ exists and $2n>N'=4i'_0$. Then, using the fact $y^n\delta z=\delta(y^n z)=0$ and (2.7.4), we have

$$\begin{aligned}\delta\rho^* z_j &= \sum_{s\in K'(j)} p_s(m\xi)\delta z_{j-s} - \sum_{s\in K'(j)} \lambda'_{j-s} y^{2j-2s-2i'_0} p_s(m\xi)\bar{f}^*_n U'_{i'_0} + p_{j-i'_0}(m\xi)\bar{f}^*_n U'_{i'_0} \\ &\quad + y^n p_{j-n}(m\xi)\delta z - \lambda'_0 y^{2n-2i'_0} p_{j-n}(m\xi)\bar{f}^*_n U'_{i'_0} \\ &\quad + \lambda_j y^{2j-2i_0}\Big\{\sum_{t\in K'(i_0)} p_t(m\xi)\delta z_{i_0-t} - \sum_{t\in K'(i_0)} \lambda'_{i_0-t} y^{2i_0-2t-2i'_0} p_t(m\xi)\bar{f}^*_n U'_{i'_0} \\ &\quad + p_{i_0-i'_0}(m\xi)\bar{f}^*_n U'_{i'_0} + y^n p_{i_0-n}(m\xi)\delta z - \lambda'_0 y^{2n-2i'_0} p_{i_0-n}(m\xi)\bar{f}^*_n U'_{i'_0}\Big\} \\ &= \delta\Big\{\sum_{s\in K'(j)} p_s(mp^*\xi)z_{j-s} + \lambda_j y^{2j-2i_0}\sum_{t\in K'(i_0)} p_t(mp^*\xi)z_{i_0-t}\Big\} + A\bar{f}^*_n U'_{i'_0},\end{aligned}$$

for some $A\in H^*(CP^\infty)$. In the same way as the proof of (3.11.1), we have $A=0$. Since $\delta$ is monomorphic in degree $4j-1$, we have

$$\rho^* z_j = \sum_{s\in K'(j)}\binom{m}{s} y^{2s} z_{j-s} + \lambda_j \sum_{t\in K'(i_0)}\binom{m}{t} y^{2j+2t-2i_0} z_{i_0-t}.$$

Assume that $N'$ exists and $2n\leq N'$ or $N'$ does not exist. Then we have

$$\begin{aligned}\delta\rho^* z_j &= \sum_{s\in K(j)} p_s(m\xi)\delta z_{j-s} + \sum_{s\in K(j)}\binom{n}{j-s} y^{2j-2s-n} p_s(m\xi)\bar{f}^*_n U' \\ &\quad + y^n p_{j-n}(m\xi)\bar{f}^*_n U' + \lambda_j y^{2j-2i_0}\Big\{\sum_{t\in K(i_0)} p_t(m\xi)\delta z_{i_0-t} \\ &\quad + \sum_{t\in K(i_0)}\binom{n}{i_0-t} y^{2i_0-2t-n} p_t(m\xi)\bar{f}^*_n U' + y^n p_{i_0-n}(m\xi)\bar{f}^*_n U'\Big\} \\ &= \delta\Big\{\sum_{s\in K(j)} p_s(mp^*\xi)z_{j-s} + \lambda_j y^{2j-2i_0}\sum_{t\in K(i_0)} p_t(mp^*\xi)z_{i_0-t}\Big\} + A'\bar{f}^*_n U',\end{aligned}$$

for some $A'\in H^*(CP^\infty)$ by (2.7.2–3). In the similar way to the above, we obtain

$$\rho^* z_j = \sum_{s\in K(j)}\binom{m}{s} y^{2s} z_{j-s} + \lambda_j \sum_{t\in K(i_0)}\binom{m}{t} y^{2j-2i_0+2t} z_{i_0-t},$$

and (5.2.3) follows.

In the similar way to the proof of (5.2.2–3), we have (5.2.4–6). Q.E.D.

REMARK. In §§3–4, we determined explicitly the reduced power operations $\mathcal{P}^i$ and the Bockstein homomorphism $\beta$ under the assumption $(*)$ of Theorem 4.12. Using the results of this section, we can expect to study $\mathcal{P}^i$ and $\beta$ for other $n$ and $k$.

## §6. Applications to the immersion problem for the lens spaces

We denote $L^n(p)$ the mod $p$ lens space of dimension $2n+1$, and $\eta_n$ the restriction of $\eta$ over $L^\infty(p)$ to $L^n(p)$. By $L^n(p)\subseteq R^k$, we mean that $L^n(p)$ can be immersed in the real $k$-space $R^k$. The next theorem for immersion was proved in [7, Theorem 1].

THEOREM 6.1 (Kobayashi). *Let $n=(p-1)s+r$ $(0\leq r<p-1)$ and $k$ be a positive integer with $k\leq 2n+1$ and let $a$ be a positive integer such that $2ap^{s+\varepsilon}>4n+3$, where $\varepsilon=0$ or $1$ according as $r\leq 1$* or $>1$. *The necessary and sufficient condition for $L^n(p)\subseteq R^{2n+1+k}$ is that the bundle $\{ap^{s+\varepsilon}-(n+1)\}\eta_n$ has $2ap^{s+\varepsilon}-(2n+k+2)$ independent cross sections.*

One of our main theorems is the following

THEOREM 6.2. *Let $r$ and $n'$ be positive integers such that $r\geq 2$ and $(p, n')=1$ and let $m$ and $t$ be non-negative integers satisfying*

$$(*)\qquad 0\leq t\leq m,\quad m-t+(p-1)/2<p^{r-1},\quad t<p^{r-1},\quad \binom{m}{t}\not\equiv 0 \bmod p.$$

*Then, the bundle $(n'p^r+m)\eta_n$ over $L^n(p)$ does not have $k$ independent cross sections for*

$$(**)\qquad k=2n'p^r-2lp^{r-1}+2t+1,\quad 2lp^{r-1}+2m-2t+p-1\leq n<2p^r,$$
$$l=1, \dots, p-1.$$

Before proving Theorem 6.2, we consider the applications.

THEOREM 6.3. *Let $r$ $(\geq 2)$, $m$ and $t$ be non-negative integers satisfying $(*)$ of Theorem 6.2, then*

$$(6.3.1)\qquad L^{p^r-m-1}(p)\not\subseteq R^{3p^r-p^{r-1}-2t-2}\qquad \text{if}\quad m\leq[(p^{r-1}-p+2t)/3],$$

$$(6.3.2)\qquad L^{2p^r-m-1}(p)\not\subseteq R^{6p^r-2p^{r-1}-2t-2}\qquad \text{if}\quad m\leq[(2p^{r-1}-p+2t)/3].$$

PROOF. Assume that $m\leq[(p^{r-1}-p+2t)/3]$ and $L^{p^r-m-1}(p)\subseteq R^{3p^r-p^{r-1}-2t-2}$. By Theorem 6.1, the bundle $(n'p^r+m)\eta_{p^r-m-1}$ has $2n'p^r-(p-1)p^{r-1}+2t+1$ independent cross sections, where $n'=ap^{s+\varepsilon-r}-1$ for some integer $a$. By the assumption $m\leq[(p^{r-1}-p+2t)/3]$, we have $(p-1)p^{r-1}+2m-2t+(p-1)\leq p^r-m-1$. This contradicts to Theorem 6.2 and so (6.3.1) follows. The proof of (6.3.2) is similar. Q.E.D.

Now, we use the following results to prove Theorem 6.2.

PROPOSITION 6.4. *Let $r$, $n'$ and $k=2k'-1$ be positive integers with $r\geq 2$, $(p, n')=1$ and $m$ be a non-negative integer such that $m<n'p^r+m-k'+1<p^r$. Then $\mathcal{P}^1$ and $\beta$ in $H^*(X_{2n'p^r+2m,k})$ are given by*

$$\mathcal{P}^1 z_j=(-1)^q(2j-1)z_{j+q}+\sum_{s=1}^{q}(-1)^{q+s}2m\, y^{2s}z_{j+q-s} \tag{6.4.1}$$

$$\textit{for}\quad n'p^r+m-k'+1\leq j<p^r-q,$$

$$\beta z_j=\sum_{l=1}^{p-1}\binom{m}{j-lp^{r-1}}\mu_l y^{2j} \qquad \textit{for}\quad n'p^r+m-k'+1\leq j<p^r, \tag{6.4.2}$$

*where* $\mu_l=\frac{1}{l}\binom{p-1}{l-1}\mu\not\equiv 0 \bmod p$ *is the same as in Lemma* 4.5 *and* $2q=p-1$.

PROOF. The homomorphism $\rho^*\colon H^*(X_{2n'p^r+2m,k})\longrightarrow H^*(X_{2n'p^r,k})$ is given by (5.2.2) if $n'\geq 3$, since $N(n'p^r+m, k)=N(n'p^r, k)=4p^r<2n'p^r$; and by (5.2.3) or (5.2.6) if $n'\leq 2$, since $N(2p^r, k)=4p^r$ and $N(p^r, k)$ does not exist. Therefore

$$\rho^* z_j=\sum_{s=0}^{m}\binom{m}{s}y^{2s}z_{j-s} \qquad \text{for}\quad n'p^r+m-k'+1\leq j<p^r,$$

since $\binom{n'p^r+m}{j}\equiv 0$ and so $\lambda_j\equiv 0 \bmod p$ for $m<j<p^r$.

Now $\mathcal{P}^1 z_j$ has the form $\mathcal{P}^1 z_j=\sum_{t=0}^{q}a_t y^{2t}z_{j+q-t}\,(a_0=(-1)^q(2j-1))$ by (3.12.1). Therefore

$$\rho^*\mathcal{P}^1 z_j=\sum_{t=0}^{q}\sum_{s=0}^{m}a_t\binom{m}{s}y^{2t+2s}z_{j+q-t-s} \qquad \text{for}\quad j+q<p^r.$$

On the other hand

$$\mathcal{P}^1\rho^* z_j=\sum_{s=0}^{m}\binom{m}{s}\{2s\, y^{2s+2q}z_{j-s}+(-1)^q(2j-2s-1)\,y^{2s}z_{j-s+q}\},$$

by (3.10.1) or (3.11.1). Comparing the coefficients of these equations, we have

$$a_0\binom{m}{s}+\cdots+a_s\binom{m}{0}\equiv(-1)^q(2j-2s-1)\binom{m}{s} \bmod p \qquad \text{for}\quad s=0, \dots, q.$$

Therefore we have $a_s=(-1)^{q+s}2m$ for $s=1, \dots, q$, by the induction on $s$ and we have (6.4.1).

If $j<p^r$, then

$$\beta\rho^* z_j=\beta\Big(\sum_{s=0}^{m}\binom{m}{s}y^{2s}z_{j-s}\Big)=\sum_{s=0}^{m}\binom{m}{s}y^{2s}\beta z_{j-s}=\sum_{l=1}^{p-1}\binom{m}{j-lp^{r-1}}\mu_l y^{2j},$$

by (4.12.1). Therefore (6.4.2) follows. Q.E.D.

LEMMA 6.5. *Suppose there is a map $f\colon L^n(p) \longrightarrow X_{2m,k}$ such that the following diagram is commutative*:

$$\begin{tikzcd} & X_{2m,k} \arrow[dr, "p"] & \\ L^n(p) \arrow[ur, "f"] & \subset & L^\infty(p). \end{tikzcd}$$

*If $2j \leq n$ and $\beta z_j = \mu y^{2j}$, then $f^* z_j = \mu x y^{2j-1}$ in $H^{4j-1}(L^n(p))$.*

PROOF. By the commutativity of the diagram, we have $f^*x = x$ and $f^*y = y$. Assume $f^*z_j = \mu' x y^{2j-1}$, then $\mu' y^{2j} = \beta f^* z_j = f^* \beta z_j = \mu y^{2j}$.
Q.E.D.

COROLLARY 6.6. *Set $m$ of Lemma 6.5 be $n'p^r + m$. Under the assumptions of Proposition 6.4, we have*

$$f^* z_j = \sum_{l=1}^{p-1} \binom{m}{j - lp^{r-1}} \mu_l x y^{2j-1} \qquad \textit{for} \quad n'p^r + m - k' + 1 \leq j < p^r.$$

PROOF OF THEOREM 6.2. Assume that $(n'p^r + m)\eta_n$ over $L^n(p)$ has $k$ independent cross sections, where $k = 2n'p^r - 2lp^{r-1} + 2t + 1$. Then its associated $V_{2n'p^r+2m,k}$-bundle has a cross section and so there exists a map $f\colon L^n(p) \longrightarrow X_{2n'p^r+2m,k}$ such that the following diagram is commutative:

$$\begin{tikzcd} & X_{2n'p^r+2m,k} \arrow[dr, "p"] & \\ L^n(p) \arrow[ur, "f"] & \subset & L^\infty(p). \end{tikzcd}$$

Let $j = lp^{r-1} + m - t$ and $2q = p - 1$. By (6.4.1), we have

$$\mathcal{P}^1 z_j = (-1)^q (2j-1) z_{j+q} + \sum_{s=1}^{q} (-1)^{q+s} 2m\, y^{2s} z_{j+q-s}. \tag{6.7}$$

Since $2(j+q) \leq n$ and $n'p^r + m - k' + 1 \leq j + q < p^r$ by the assumption $(**)$, $f^* z_{j+q-s}$ $(0 \leq s \leq q)$ is given by Corollary 6.6, and its coefficient is $\sum_{l'=1}^{p-1} \binom{m}{(l-l')p^{r-1} + q - s + m - t} \mu_{l'}$. In this summation, the binomial coefficients are zero if $l' \neq l$ by the condition $(*)$. Therefore we have

$$f^* z_{j+q-s} = \binom{m}{t - q + s} \mu_l x y^{2j+2q-2s-1} \qquad \text{for} \quad 0 \leq s \leq q. \tag{6.8}$$

If $0 \leq t \leq q-1$, we have

$$f^* \mathcal{P}^1 z_j = 2m \left\{ \sum_{s=q-t}^{q} (-1)^{q+s} \binom{m}{t - q + s} \mu_l \right\} x y^{2j+2q-1}$$

$$= 2m \binom{m-1}{t} \mu_l x y^{2j+2q-1} = (2m - 2t) \binom{m}{t} \mu_l x y^{2j+2q-1},$$

by (6.7–8) and the simple calculations of the binomial coefficients. On the other hand, we obtain

$$\mathcal{P}^1 f^* z_j = \binom{m}{t}\mu_l \mathcal{P}^1(x\,y^{2j-1}) = \binom{m}{t}(2j-1)\mu_l x\,y^{2j+2q-1}$$
$$= (2m-2t-1)\binom{m}{t}\mu_l x\,y^{2j+2q-1},$$

by (6.8). Since $\binom{m}{t} \not\equiv 0 \bmod p$ and $\mu_l \not\equiv 0 \bmod p$, we have $f^*\mathcal{P}^1 z_j \not\equiv \mathcal{P}^1 f^* z_j$, which is a contradiction.

If $t=q$, we obtain similarly

$$f^*\mathcal{P}^1 z_j = \left\{(-1)^q(2j-1) + \sum_{s=1}^{q}(-1)^{q+s}2m\binom{m}{s}\right\}\mu_l x\,y^{2j+2q-1}$$
$$= (2m-1)\binom{m}{q}\mu_l x\,y^{2j+2q-1},$$
$$\mathcal{P}^1 f^* z_j = 2m\binom{m}{q}\mu_l x\,y^{2j+2q-1},$$

which is a contradiction.

Finally, if $t>q$, we have similarly a contradiction:

$$f^*\mathcal{P}^1 z_j = (2m-2t)\binom{m}{t}\mu_l x\,y^{2j+2q-1},$$
$$\mathcal{P}^1 f^* z_j = (2m-2t-1)\binom{m}{t}\mu_l x\,y^{2j+2q-1}. \qquad \text{Q.E.D.}$$

REMARK 6.9. *Comparing Theorem* 6.3 *with D. Sjerve's Theorem for immersions* [14, Theorem 4.7 (i)], *we have, e.g., the following results*:

$$L^n(p) \not\subseteq R^{3n-p+1}, \qquad L^n(p) \subseteq R^{3n-p+3}$$

*if*

$$n = n'p^r - [(n'p^{r-1} - p + 2t)/3] - 1$$
$$= n'p^r - (n'p^{r-1} - p + 2t)/3 - 1,\ n' = 1 \text{ or } 2;$$

$$L^n(p) \not\subseteq R^{3n-p}, \qquad L^n(p) \subseteq R^{3n-p+4}$$

*if*

$$n = n'p^r - [(n'p^{r-1} - p + 2t)/3] - 1$$
$$= n'p^r - (n'p^{r-1} - p + 2t - 1)/3 - 1,\ n' = 1 \text{ or } 2.$$

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*Department of Mathematics,*
*Faculty of Science,*
*Hiroshima University*