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Note on the Enumeration of Embeddings of Real Projective Spaces, II

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Introduction

In the previous note [19], under the same title we studied the enumeration problem of embeddings of the n-dimensional real projective space RP^n in the real $(2n-2)$-space R^{2n-2} for even n. In this note, we shall study this problem for odd n and prove the following

THEOREM C. *Let $n\equiv 1(4)$, $n\neq 2^r+1$ and let $n\geq 13$. Then there are eight distinct isotopy classes of embeddings of RP^n in R^{2n-2}.*

To prove this theorem by applying [19, § 5, Proposition], we shall calculate the cohomology group of the reduced symmetric product $(RP^n)^*$ of RP^n for odd n in § 8.

As for the case $n\equiv 3(4)$, we now notice the following result in § 10.

PROPOSITION D. *Let $n\equiv 3(4)$ and $n\geq 11$. Then*

$$16\leq \#[RP^n\subset R^{2n-2}]\leq 32, \qquad \#[RP^n\subset R^{2n-2}]\equiv 0(4),$$

where $\#[RP^n\subset R^{2n-2}]$ denotes the cardinality of the set of isotopy classes of embeddings of RP^n in R^{2n-2}.

We shall freely use the notations in [19].

§ 8. Remarks on the cohomology of $(RP^n)^*$ for odd n

According to [7, (2.5–6)], there is a commutative diagram of double coverings

$$\begin{array}{ccc} V_{n+1,2}/(Z_2+Z_2)=Z_{n+1,2} & \xrightarrow{f'} & RP^n\times RP^n-\Delta \\ \downarrow & & \downarrow \\ V_{n+1,2}/D_4=SZ_{n+1,2} & \xrightarrow{f} & (RP^n)^*, \end{array}$$

where $V_{n+1,2}$ is the Stiefel manifold of 2-frames in R^{n+1}, D_4 is the dihedral group of order 8, both f and f' are homotopy equivalences and both $Z_{n+1,2}$ and $SZ_{n+1,2}$

are $(2n-1)$-dimensional manifolds.

(8.1) *For odd n, the integral cohomology group* $H^i(Z_{n+1,2}; Z)=H^i(RP^n\times RP^n-\Delta; Z)$ $(i\geq 1)$ *is finite and has no odd torsion.*

PROOF. Since n is odd, RP^n is orientable and so is $RP^n\times RP^n$. The Poincaré-Lefschetz duality provides the isomorphism $H^{2n-i}(RP^n\times RP^n-\Delta; Z)=H_i(RP^n\times RP^n, \Delta; Z)$ for all i. This isomorphism and the split short exact sequence $0\to H_i(RP^n; Z)\to H_i(RP^n\times RP^n; Z)\to H_i(RP^n\times RP^n, \Delta; Z)\to 0$ yield (8.1).

Let $\underline{Z}=\{Z\}$ be the local system on $SZ_{n+1,2}$ associated with the double covering $Z_{n+1,2}\to SZ_{n+1,2}$, and consider the two Thom-Gysin exact sequences ([16, pp. 282–283]) associated with this double covering:

$$\cdots\to H^i(SZ_{n+1,2}; Z)\to H^i(Z_{n+1,2}; Z)\to H^i(SZ_{n+1,2}; \underline{Z})\to H^{i+1}(SZ_{n+1,2}; Z)\to\cdots,$$

$$\cdots\to H^i(SZ_{n+1,2}; \underline{Z})\to H^i(Z_{n+1,2}; Z)\to H^i(SZ_{n+1,2}; Z)\to H^{i+1}(SZ_{n+1,2}; \underline{Z})\to\cdots.$$

By using these exact sequences and (8.1), we see the following result by induction.
(8.2) *For odd n,* $H^i(SZ_{n+1,2}; \underline{Z})$ *and* $H^i(SZ_{n+1,2}; Z)=H^i((RP^n)^*; Z)$ *are finite and have no odd torsion.*

Now, let $n=2^r+s$ (≥ 11), $0<s<2^r$ and s be odd. Then (6.3) also holds by the same proof as in § 6, that is,
(8.3) *the* mod 2 *cohomology group* $H^i((RP^n)^*; Z_2)$ *for* $2n-4\leq i\leq 2n-1$ *is given as follows*:

i	$H^i((RP^n)^*; Z_2)$	*basis*
$2n-1$	Z_2	$vx^{2^{r+1}-2}y^s$
$2n-2$	Z_2+Z_2	$vx^{2^{r+1}-3}y^s,\ x^{2^{r+1}-2}y^s$
$2n-3$	$Z_2+Z_2+Z_2$	$vx^{2^{r+1}-4}y^s,\ x^{2^{r+1}-3}y^s,\ vx^{2^{r+1}-2}y^{s-1}$
$2n-4$	$Z_2+Z_2+Z_2+Z_2$	$vx^{2^{r+1}-5}y^s,\ x^{2^{r+1}-4}y^s,\ vx^{2^{r+1}-3}y^{s-1},\ x^{2^{r+1}-2}y^{s-1}$

where $\deg v=\deg x=1$, $\deg y=2$, $v^2=vx$, $Sq^1y=xy$ *and* $x^{2^{r+1}-1}=0$.

Furthermore, by the result of S. Feder [5, Corollary 4.1] and (6.1),
(8.4) $x^{2i}y^{n-i-1}\neq 0$ *if and only if* $i=2^t-1$ *for some t.*

Since s is odd, simple calculations show the relations

$$Sq^1(vx^{2^{r+1}-5}y^s)=vx^{2^{r+1}-4}y^s,\quad Sq^1(x^{2^{r+1}-4}y^s)=x^{2^{r+1}-3}y^s,$$

$$vx^{2^{r+1}-3}y^{s-1} = Sq^1(vx^{2^{r+1}-4}y^{s-1}), \quad x^{2^{r+1}-2}y^{s-1} = Sq^1(x^{2^{r+1}-3}y^{s-1}).$$

Consider the Bockstein exact sequence

$$\cdots \longrightarrow H^{2n-4}((RP^n)^*; Z) \xrightarrow{\rho_2} H^{2n-4}((RP^n)^*; Z_2) \xrightarrow{\beta_2} H^{2n-3}((RP^n)^*; Z)$$
$$\xrightarrow{\times 2} H^{2n-3}((RP^n)^*; Z) \xrightarrow{\rho_2} H^{2n-3}((RP^n)^*; Z_2) \longrightarrow \cdots$$

associated with $0 \longrightarrow Z \xrightarrow{\times 2} Z \xrightarrow{\rho_2} Z_2 \longrightarrow 0$. Then (8.2), (8.3) and the above relations for $Sq^1 = \rho_2\beta_2$ yield the following results:
(8.5) $\rho_2 H^{2n-4}((RP^n)^*; Z) = Z_2 + Z_2$ *generated by* $\{vx^{2^{r+1}-3}y^{s-1}, x^{2^{r+1}-2}y^{s-1}\}$ *and* $H^{2n-3}((RP^n)^*; Z) = Z_2 + Z_2$ *generated by* $\{\beta_2(x^{2^{r+1}-4}y^s), \beta_2(vx^{2^{r+1}-5}y^s)\}$.

§9. Proof of Theorem C

Now, we prove the following

THEOREM C. *Let* $n \equiv 1(4)$, $n \neq 2^r+1$ *and let* $n \geq 13$. *Then*

$$\#[RP^n \subset R^{2n-2}] = 8.$$

PROOF. The existence of an embedding of RP^n in R^{2n-2} is shown in [10, Theorem 7.2.2].

Consider the proposition in §5 for $M = RP^n$, where the homomorphisms
Θ^i: $H^{i-1}((RP^n)^*; Z) \longrightarrow H^{i+1}((RP^n)^*; Z_2)$ for $i = 2n-2, 2n-3$,

$$\Gamma: H^{2n-3}((RP^n)^*; Z_2) \longrightarrow H^{2n-1}((RP^n)^*; Z_2)$$

are given by $\Theta^i(a) = Sq^2\rho_2 a$, $\Gamma(b) = Sq^2 b$ because n is odd.

Let $n = 2^r + s$, $0 < s < 2^r$. By the relations in (8.3), simple calculations show that $Sq^2(y^t) = ty^{t+1} + \binom{t}{2}x^2y^t$, and so we have $\Gamma(vx^{2^{r+1}-4}y^s) = Sq^2(vx^{2^{r+1}-4}y^s) = vx^{2^{r+1}-2}y^s + \binom{s}{2}vx^{2^{r+1}-2}y^s = vx^{2^{r+1}-2}y^s$ by (8.4) and the assumption that $s \equiv 1(4)$. Therefore, by (8.3),

(9.1) Γ *is an epimorphism.*

Also, by the relations in (8.3) and (8.4), we see easily that

$$\Theta^{2n-2}\beta_2(vx^{2^{r+1}-5}y^s) = vx^{2^{r+1}-2}y^s, \ \Theta^{2n-2}\beta_2(x^{2^{r+1}-4}y^s) = 0,$$

since $\Theta^{2n-2}\beta_2 = Sq^2Sq^1$. These relations, (8.3) and (8.5) show that

(9.2) $$\operatorname{Ker}\Theta^{2n-2} = Z_2.$$

Furthermore, we see easily that

$$Sq^2(x^{2^{r+1}-2}y^{s-1}) = Sq^2(vx^{2^{r+1}-3}y^{s-1}) = 0$$

by the relations in (8.3). Therefore, by (8.5), we have

(9.3) $$\operatorname{Coker}\Theta^{2n-3} = H^{2n-2}((RP^n)^*; Z_2) = Z_2+Z_2.$$

By (9.1)-(9.3), Theorem C follows from the proposition in §5 for $M=RP^n$.

§10. Proof of Proposition D

Finally, we notice the following

PROPOSITION D. *Let* $n\equiv 3(4)$ *and* $n\geq 11$. *Then*

$$16 \leq \#[RP^n \subset R^{2n-2}] \leq 32, \quad \#[RP^n \subset R^{2n-2}] \equiv 0(4).$$

PROOF. The existence of an embedding of RP^n in R^{2n-2} is shown in [10, Theorem 7.2.2].

By Y. Nomura's theorem [12, Theorem 2.4], we have

(10.1) $$[RP^n \subset R^{2n-2}] = \bigcup_{\sigma\in\operatorname{Ker}\Theta^{2n-2}} (H^{2n-2}((RP^n)^*; Z_2)/\operatorname{Im}\Theta^{2n-3}) \times \operatorname{Coker}\Phi_\sigma,$$

where Φ_σ: $\operatorname{Ker}\Theta^{2n-3}\to\operatorname{Coker}\Gamma$ is the twisted secondary operation defined in [12, §2, p. 6] and Θ^i $(i=2n-2, 2n-3)$ and Γ are the homomorphisms given in the proof of §9.

On the other hand, we have the following relations by the similar calculations to those in §9 noticing that $s\equiv 3(4)$:

$$Sq^2(vx^{2^{r+1}-3}y^{s-1}) = Sq^2(x^{2^{r+1}-2}y^{s-1}) = 0,$$

$$\Theta^{2n-2}\beta_2(vx^{2^{r+1}-5}y^s) = \Theta^{2n-2}\beta_2(x^{2^{r+1}-4}y^s) = 0,$$

$$\Gamma(vx^{2^{r+1}-4}y^s) = \Gamma(x^{2^{r+1}-3}y^s) = \Gamma(vx^{2^{r+1}-2}y^{s-1}) = 0.$$

Therefore, it follows from (8.3) and (8.5) that

$$H^{2n-2}((RP^n)^*; Z_2)/\operatorname{Im}\Theta^{2n-3} = Z_2+Z_2,$$

$$\operatorname{Ker}\Theta^{2n-2} = Z_2+Z_2, \quad \operatorname{Coker}\Gamma = Z_2.$$

Hence $\operatorname{Coker}\Phi_\sigma = 0$ or Z_2 for any $\sigma\in\operatorname{Ker}\Theta^{2n-2}$, and so we have Proposition D by (10.1).

References

(continued from [19])

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