HIROSHIMA MATH. J.
8 (1978), 235-253

The Enumeration of Embeddings of Lens Spaces and Projective Spaces

Tsutomu YASUI

(Received November 1, 1977)

Introduction

The purpose of this article is to study the enumeration problem of embeddings of the lens space $L^n(p)$ mod p (odd prime), the real projective space RP^n and the complex projective space CP^n in Euclidean spaces.

Let M be an m-dimensional closed differentiable manifold, and let $g: M^* \to RP^\infty$ (the infinite dimensional real projective space) denote the classifying map of the double covering

$$\pi: M \times M - \Delta \longrightarrow M^* = (M \times M - \Delta)/Z_2$$

over the reduced symmetric product M^* of M, where Δ is the diagonal and Z_2 acts on $M \times M - \Delta$ via $t(x, y) = (y, x)$. Also Z_2 acts on the n-dimensional sphere S^n via the antipodal map and we obtain the fiber bundle

$$p: (S^\infty \times S^n)/Z_2 \ (\simeq RP^n) \longrightarrow RP^\infty$$

which is homotopically equivalent to the natural inclusion $RP^n \subset RP^\infty$. Then the following theorem is due to A. Haefliger [7].

THEOREM. *Let $2(n+1) > 3(m+1)$. If there exists an embedding of M in R^{n+1}, then there exists a bijection between the set $[M \subset R^{n+1}]$ of isotopy classes of embeddings of M in R^{n+1} and the set $[M^*, RP^n; g]$ of (vertical) homotopy classes of liftings of $g: M^* \to RP^\infty$ to RP^n.*

The set $[M^*, RP^n; g]$ has the structure of an abelian group by J. C. Becker [2]. Thus, the set $[M \subset R^{n+1}]$ is an abelian group via the bijection of this theorem. We study the groups $[L^n(p) \subset R^{4n+2-i}]$, $[RP^n \subset R^{2n-i}]$ and $[CP^n \subset R^{4n-i}]$ for $i < 6$ and prove the theorems below.

THEOREM A. *The following statements hold for odd prime p:*

(1) $[L^n(p) \subset R^{4n+1}] = 0$, $\quad n > 2$.

(2) $[L^n(p) \subset R^{4n}] = Z_p$, $\quad n > 3$.

(3) $[L^n(p) \subset R^{4n-1}] = Z_p$, $\quad n > 4$.

$$(4)\quad [L^n(p) \subset R^{4n-2}] = \begin{cases} Z_p + Z_p, & p \neq 3,\ n > 5, \\ Z_3 + Z_3 + Z_9, & p = 3,\ n \equiv 2(3),\ n > 5, \\ Z_9, & p = 3,\ n \not\equiv 2(3),\ n > 5. \end{cases}$$

$$(5)\quad [L^n(p) \subset R^{4n-3}] = Z_p, \qquad n > 6.$$

THEOREM B. *The following statements hold for even n:*

(1) *Let $n \geq 10$. If there is an embedding of RP^n in R^{2n-3}, then*

$$[RP^n \subset R^{2n-3}] = \begin{cases} Z_2, & n \not\equiv 6(8), \\ Z_2 + Z_2, & n \equiv 6(8). \end{cases}$$

(2) *Let $n \geq 12$. If there is an embedding of RP^n in R^{2n-4}, then*

$$[RP^n \subset R^{2n-4}] = \begin{cases} 0, & n \equiv 0(4), \\ Z_2, & n \equiv 2(8), \\ Z_2 + Z_2 + Z_2, & n \equiv 6(8). \end{cases}$$

(3) *Let $n \geq 12$. If there is an embedding of RP^n in R^{2n-5}, then*

$$[RP^n \subset R^{2n-5}] = Z_2, \qquad n \equiv 0(4),$$

$$\#[RP^n \subset R^{2n-5}] = \begin{cases} 4, & n \equiv 2(8), \\ 8 \text{ or } 16, & n \equiv 6(8), \end{cases}$$

where $\#S$ denotes the cardinality of the set S.

THEOREM C. *The following statements hold:*

(1) *Let $n > 5$, $n \neq 2^r + 2^s$ $(r \geq s > 0)$. Then*

$$[CP^n \subset R^{4n-3}] = \begin{cases} Z, & n \equiv 0(2), \\ Z + Z_2, & n \equiv 1(2). \end{cases}$$

(2) *Let $n > 6$. If there is an embedding of CP^n in R^{4n-4}, then*

$$[CP^n \subset R^{4n-4}] = 0, \qquad n \equiv 0(2).$$

(3) *Let $n > 7$. If there is an embedding of CP^n in R^{4n-5}, then*

$$[CP^n \subset R^{4n-5}] = Z + Z, \qquad n \equiv 0(2).$$

For the assumptions of the existence of an embedding in Theorems B and C, there are several known results, cf. e.g., [14] and [16]. By this time, D. R.

Bausum, L. L. Larmore, R. D. Rigdon and the author have studied $[RP^n \subset R^{2n-i}]$ for $i<3$ and $[CP^n \subset R^{4n-i}]$ for $i<3$ in [1], [9], [19], [20] and [18].

We devote § 1 to the construction of a finite decreasing filtration of the group $[X, RP^n; f]$ of homotopy classes of liftings of $f: X \to RP^\infty$ to RP^n. Next, we calculate the cohomology of $L^n(p)^*$ in §2 and prove Theorem A in § 3. In § 4, we calculate the cohomology of $(RP^n)^*$ and $(CP^n)^*$ and in § 5, we prove Theorems B and C.

§ 1. Enumeration of liftings in the fibration $RP^n \to RP^\infty$

D. R. Bausum constructed in [1, §§ 1–3] the fifth stage Postnikov factorization of the fibration $p: RP^n \to RP^\infty$ with fiber S^n and converted it into the factorization of the fibration $(RP^n)^2 \to RP^n$ which is the pullback of p by p. However, we use a somewhat modified factorization given as follows ($n \geq 8$):

$$\begin{array}{ccccccccccc} & & & & C_3 & & C_2 & & C_1 & & \\ & & & & \uparrow & & \uparrow & & \uparrow & & \\ (RP^n)^2 & \xrightarrow{q} & E_4 & \xrightarrow{p_4} & E_3 & \xrightarrow{p_3} & E_2 & \xrightarrow{p_2} & E_1 & \longrightarrow & RP^n, \end{array}$$

$$E_1 = \begin{cases} K(Z, n) \times RP^n, & n \equiv 1(2), \\ L_\phi(Z, n) \times_{RP^\infty} RP^n, & n \equiv 0(2), \end{cases}$$

$$C_1 = \begin{cases} K(Z_2, n+2) \times K(Z_2, n+4) \times K(Z_3, n+4) \times RP^n, & n \equiv 1(2), \\ K(Z_2, n+2) \times K(Z_2, n+4) \times L_{\phi'}(Z_3, n+4) \times_{RP^\infty} RP^n, & n \equiv 0(2), \end{cases}$$

$$C_2 = K(Z_2, n+3) \times K(Z_2, n+4) \times RP^n,$$

$$C_3 = K(Z_2, n+4) \times RP^n,$$

and the map q is an $(n+6)$-equivalence. Here $L_\phi(Z, n) \times_{RP^\infty} RP^n$ is the pullback of $L_\phi(Z, n) = S^\infty \times_{Z_2} K(Z, n)^{*)} \to S^\infty/Z_2 = RP^\infty$ by $p: RP^n \to RP^\infty$, where the action of Z_2 on $K(Z, n)$ is induced from the non-trivial homomorphism $\phi: Z_2 \to \mathrm{Aut}(Z)$. Also $L_{\phi'}(Z_3, n+4) \times_{RP^\infty} RP^{n*)}$ is defined in the same way by using the non-trivial homomorphism $\phi': Z_2 \to \mathrm{Aut}(Z_3)$.

Let X be a CW-complex of dimension less than $n+6$ and let $n>7$. If $g: X \to RP^\infty$ has a lifting f to RP^n, then $[X, RP^n; g] \approx [X, (RP^n)^2; f]$. By the standard exact couple argument, we can construct a spectral sequence. In this spectral

*) $L_\phi(Z, n) = K(Z, n; \phi)$ and $L_{\phi'}(Z_3, n+4) = K(Z_3, n+4; \phi')$ by Bausum's notation.

sequence, the differentials d_1 are given by the following primary operations:

Case I. $n \equiv 1(2)$.

$$\Theta^i: H^{i-1}(X; Z) \longrightarrow H^{i+1}(X; Z_2) \times H^{i+3}(X; Z_2) \times H^{i+3}(X; Z_3),$$

$$\Theta^i(a) = (Sq^2\rho_2 a + \varepsilon_1 v^2 \rho_2 a,\ Sq^4 \rho_2 a + \varepsilon_2 v^4 \rho_2 a,\ \mathscr{P}_3^1 \rho_3 a);$$

$$\Gamma^i: H^i(X; Z_2) \times H^{i+2}(X; Z_2) \times H^{i+2}(X; Z_3) \longrightarrow H^{i+2}(X; Z_2) \times H^{i+3}(X; Z_2),$$

$$\Gamma^i(a, b, c) = (Sq^2 a + \varepsilon_1 v^2 a,\ Sq^2 Sq^1 a + Sq^1 b);$$

$$\Delta^i: H^{i+1}(X; Z_2) \times H^{i+2}(X; Z_2) \longrightarrow H^{i+3}(X; Z_2),$$

$$\Delta^i(a, b) = Sq^2 a + \varepsilon_1 v^2 a + Sq^1 b;$$

where

$$\varepsilon_1 = \begin{cases} 1, & n \equiv 1(4), \\ 0, & n \equiv 3(4), \end{cases} \qquad \varepsilon_2 = \begin{cases} 1, & n \equiv 3, 5(8), \\ 0, & n \equiv 1, 7(8). \end{cases}$$

Case II. $n \equiv 0(2)$.

$$\Theta^i: H^{i-1}(X; \underline{Z}) \longrightarrow H^{i+1}(X; Z_2) \times H^{i+3}(X; Z_2) \times H^{i+3}(X; \underline{Z}_3),$$

$$\Theta^i(a) = (Sq^2\rho_2 a + \varepsilon_3 v^2 \rho_2 a,\ Sq^4 \rho_2 a + \varepsilon_4 v^4 \rho_2 a,\ \mathscr{P}_3^1 \rho_3 a),$$

($\mathscr{P}_3^1$ is the reduced power operation mod 3 in local coefficients [6]);

$$\Gamma^i: H^i(X; Z_2) \times H^{i+2}(X; Z_2) \times H^{i+2}(X; \underline{Z}_3) \longrightarrow H^{i+2}(X; Z_2) \times H^{i+3}(X; Z_2),$$

$$\Gamma^i(a, b, c) = ((Sq^2 + vSq^1 + (1-\varepsilon_3)v^2)a,\ (Sq^2 Sq^1 + v^2 Sq^1 + \varepsilon_3 v^3)a + (Sq^1 + v)b);$$

$$\Delta^i: H^{i+1}(X; Z_2) \times H^{i+2}(X; Z_2) \longrightarrow H^{i+3}(X; Z_2),$$

$$\Delta^i(a, b) = Sq^2 a + (1-\varepsilon_3)v^2 a + Sq^1 b + vb;$$

$$\varepsilon_3 = \begin{cases} 1, & n \equiv 2(4), \\ 0, & n \equiv 0(4), \end{cases} \qquad \varepsilon_4 = \begin{cases} 1, & n \equiv 4, 6(8), \\ 0, & n \equiv 0, 2(8). \end{cases}$$

In Cases I and II, ρ_p is the mod p reduction, $v = g^* z$, where z is the generator of $H^1(RP^\infty; Z_2) = Z_2$, and $\underline{Z}$ and $\underline{Z}_3$ are the local systems on X induced by $\pi_1(X)$

$\xrightarrow{g_*} \pi_1(RP^\infty)=Z_2 \xrightarrow{\phi} \mathrm{Aut}(Z)$ and $\pi_1(X) \xrightarrow{g_*} Z_2 \xrightarrow{\phi'} \mathrm{Aut}(Z_3)$, respectively.
Further, the differentials d_2 are given by the secondary operations

$$\Phi^i: \mathrm{Ker}\,\Theta^i \longrightarrow \mathrm{Ker}\,\Delta^{i+1}/\mathrm{Im}\,\Gamma^i,$$

$$\Psi^i: \mathrm{Ker}\,\Gamma^i/\mathrm{Im}\,\Theta^{i-1} \longrightarrow \mathrm{Coker}\,\Delta^i,$$

defined by $\Gamma^{i+1}\Theta^i=0$ and $\Delta^{i+1}\Gamma^i=0$. Also, the differential d_3 is a tertiary operation

$$\chi^i: \mathrm{Ker}\,\Phi^i \longrightarrow \mathrm{Coker}\,\Psi^i.$$

Then the theorem of J. C. McClendon [12, Theorem 5.1] is stated as follows:

PROPOSITION 1.1. *Let X be a CW-complex of dimension less than $n+6$ and let $n>7$. If $g: X\to RP^\infty$ has a lifting to RP^n, then*
(1) *$[X, RP^n; g]$ has a natural abelian group structure and*
(2) *there exists a decreasing filtration of $[X, RP^\infty; g]$:*

$$[X, RP^n; g] = F_0 \supset F_1 \supset F_2 \supset F_3 \supset 0,$$

such that

$$F_0/F_1 = \mathrm{Ker}\,\chi^{n+1}, \qquad F_1/F_2 = \mathrm{Ker}\,\Psi^{n+1},$$

$$F_2/F_3 = \mathrm{Coker}\,\Phi^n, \qquad F_3 = \mathrm{Coker}\,\chi^n.$$

§2. The cohomology of $L^n(p)^*$

The purpose of this section is to study the cohomology groups $H^i(L^n(p)^*; G)$ of the reduced symmetric product $L^n(p)^*$ of the lens space $L^n(p)$ mod p, where p is an odd prime. Here the coefficient G is either Z, Z_2, Z_3 or the local systems $\underline{Z}$, $\underline{Z}_3$ induced from the double covering $\pi: L^n(p)\times L^n(p)-\Delta\to L^n(p)^*$. We always use the Bockstein exact sequences

$$\begin{aligned}(2.1)\quad &\cdots \longrightarrow H^{i-1}(\ ;Z_q) \xrightarrow{\delta_q} H^i(\ ;Z) \xrightarrow{\times q} H^i(\ ;Z) \xrightarrow{\rho_q} H^i(\ ;Z_q) \longrightarrow \cdots,\\ &\cdots \longrightarrow H^{i-1}(\ ;\underline{Z}_q) \xrightarrow{\delta_q} H^i(\ ;\underline{Z}) \xrightarrow{\times q} H^i(\ ;\underline{Z}) \xrightarrow{\rho_q} H^i(\ ;\underline{Z}_q) \longrightarrow \cdots,\end{aligned}$$

associated with $0\to Z \xrightarrow{\times q} Z \xrightarrow{\rho_q} Z_q\to 0$.

Let x and y be the generators of $H^2(L^n(p); Z)=Z_p$ and $H^1(L^n(p); Z_p)=Z_p$, respectively, such that $\delta_p y=x$. Denote $\rho_p x$ by the same symbol x. Then the mod p cohomology ring of $L^n(p)$ is given by

$$(2.2)\qquad H^*(L^n(p); Z_p) = \Lambda(y)\otimes Z_p[x]/(x^{n+1}),$$

where $\Lambda(y)$ denotes the exterior algebra on y; and the integral cohomology is given by

$$(2.3)\quad H^i(L^n(p); Z) = \begin{cases} Z, & i = 0, 2n+1, \\ Z_p \textit{ generated by } x^{i/2}, & i \equiv 0(2),\ 0 < i \leq 2n, \\ 0, & \textit{otherwise}, \end{cases}$$

where $H^{2n+1}(L^n(p); Z)$ is generated by the cohomology fundamental class $[L^n(p)]$, and the relation $\rho_p[L^n(p)] = yx^n$ holds.

The next lemma is an immediate result of [16, Proposition 2.9] and (2.2–3).

LEMMA 2.4. *The* mod 2 *cohomology groups of $L^n(p)^*$ are given by*

$$H^i(L^n(p)^*; Z_2) = \begin{cases} Z_2 & \textit{for} \quad 0 \leq i \leq 2n+1, \\ 0 & \textit{otherwise}. \end{cases}$$

COROLLARY 2.5. *The cohomology groups $H^i(L^n(p)^*; Z)$ and $H^i(L^n(p)^*; \underline{Z})$ are finite and have no 2-torsions for $i > 2n+1$.*

For an automorphism σ of the group G, G^σ denotes the subgroup of the invariant elements with respect to σ. By using this corollary, the applications of the Serre spectral sequence of the fibration $L^n(p) \times L^n(p) - \Delta \xrightarrow{\pi} L^n(p)^* \to RP^\infty$ and its twisted version (see [12, § 1]) show the following

LEMMA 2.6. *Both homomorphisms*

$$\pi^*: H^i(L^n(p)^*; Z \ (or\ Z_3)) \longrightarrow H^i(L^n(p) \times L^n(p) - \Delta; Z \ (or\ Z_3))^{t^*} \qquad for \quad i > 2n+1,$$

$$\pi^*: H^i(L^n(p)^*; \underline{Z} \ (or\ \underline{Z}_3)) \longrightarrow H^i(L^n(p) \times L^n(p) - \Delta; Z \ (or\ Z_3))^{-t^*} \qquad for \quad i > 2n+1,$$

are isomorphisms, where t is the involution transposing the factors.

Hereafter we identify $H^i(L^n(p)^*; Z)$ and $H^i(L^n(p)^*; \underline{Z})$ with $H^i(L^n(p) \times L^n(p) - \Delta; Z)^{t^*}$ and $H^i(L^n(p) \times L^n(p) - \Delta; Z)^{-t^*}$ for $i > 2n+1$, respectively. Consider the Thom isomorphism

$$\phi: H^i(L^n(p); Z) \xrightarrow{\cong} H^{2n+1+i}(L^n(p) \times L^n(p), L^n(p) \times L^n(p) - \Delta; Z),$$

$$\phi(x^j) = U \cup (1 \times x^j), \quad \text{if} \quad 2j = i,\ 0 < j \leq n,$$

where the Thom class $U \in H^{2n+1}(L^n(p) \times L^n(p), L^n(p) \times L^n(p) - \Delta; Z) = Z$ is the generator. The Thom isomorphism and the cohomology exact sequence of the pair $(L^n(p) \times L^n(p), L^n(p) \times L^n(p) - \Delta)$ lead to the following

LEMMA 2.7. *The homomorphism*

$$i^*: H^{2k}(L^n(p) \times L^n(p); Z) \longrightarrow H^{2k}(L^n(p) \times L^n(p) - \Delta; Z), \quad 4n+2 > 2k > 2n+1,$$

is an isomorphism and the sequence

$$0 \longrightarrow Z_p \xrightarrow{j^*} H^{2k+1}(L^n(p) \times L^n(p); Z) \xrightarrow{i^*} H^{2k+1}(L^n(p) \times L^n(p) - \Delta; Z) \longrightarrow 0, \quad 2k+1 > 2n+1,$$

is exact, where i and j are the natural inclusions.

Moreover, the action of t^* on $H^*(L^n(p) \times L^n(p), L^n(p) \times L^n(p) - \Delta; Z)$ is well-known [15, p. 305], and is given by

$$(2.8) \qquad t^*a = -a \quad \text{for} \quad a \in H^*(L^n(p) \times L^n(p), L^n(p) \times L^n(p) - \Delta; Z).$$

LEMMA 2.9. *For* $i < 2n+1$,

$$H^{4n+2-i}(L^n(p) \times L^n(p); Z)/\mathrm{Ker}\, i^* = \begin{cases} Z_p^{2j} & \text{for} \quad i = 4j, \\ Z_p^{2j+1} & \text{for} \quad i = 4j+1, 4j+2, \\ Z_p^{2j+2} & \text{for} \quad i = 4j+3, \end{cases}$$

(G^k denotes the direct sum of k-copies of G), generated by the set $A \cup B$ given as follows:

$$A = \begin{cases} \{x^{n-k} \times x^{n+1-2j+k} + x^{n+1-2j+k} \times x^{n-k} \mid 0 \le k \le j-1\}, & i = 4j, \\ \{x^{n-k} \times x^{n-2j+k} + x^{n-2j+k} \times x^{n-k}, x^{n-j} \times x^{n-j} \mid 0 \le k \le j-1\}, & i = 4j+2, \\ \{\delta_p(yx^{n-k} \times yx^{n-2j-1+k} - yx^{n-2j-1+k} \times yx^{n-k}) \mid 0 \le k \le j\}, & i = 4j+1, \\ \{\delta_p(yx^{n-k} \times yx^{n-2j-2+k} - yx^{n-2j-2+k} \times yx^{n-k}) \mid 0 \le k \le j\}, & i = 4j+3; \end{cases}$$

$$B = \begin{cases} \{x^{n-k} \times x^{n+1-2j+k} - x^{n+1-2j+k} \times x^{n-k} \mid 0 \le k \le j-1\}, & i = 4j, \\ \{x^{n-k} \times x^{n-2j+k} - x^{n-2j+k} \times x^{n-k} \mid 0 \le k \le j-1\}, & i = 4j+2, \\ \{\delta_p(yx^{n-k} \times yx^{n-2j-1+k} + yx^{n-2j-1+k} \times yx^{n-k}) \mid 1 \le k \le j\}, & i = 4j+1, \\ \{\delta_p(yx^{n-k} \times yx^{n-2j-2+k} + yx^{n-2j-2+k} \times yx^{n-k}), \\ \quad \delta_p(yx^{n-j-1} \times yx^{n-j-1}) \mid 1 \le k \le j\}, & i = 4j+3. \end{cases}$$

If we notice that

$$j^*U = \pm(1 \times [L^n(p)] - [L^n(p)] \times 1 + \sum_{i=1}^{[n/2]} \delta_p(yx^{n-i} \times yx^{i-1} + yx^{i-1} \times yx^{n-i}) + \{\delta_p(yx^{[n/2]} \times yx^{[n/2]})\}),$$

(the term in the bracket { } is present only when n is odd), then the proof of this lemma is a simple calculation.

By identifying $H^{4n+2-i}(L^n(p) \times L^n(p); Z)/\mathrm{Ker}\, i^*$ with $H^{4n+2-i}(L^n(p) \times L^n(p) - \Delta; Z)$ by i^* for $i<2n+1$, the integral cohomology group and the cohomology group with coefficients in $\underline{Z}$ of $L^n(p)^*$ are determined by Lemmas 2.6–9.

PROPOSITION 2.10. *Let $i<2n+1$. Then*

$$H^{4n+2-i}(L^n(p)^*; Z) = \begin{cases} Z_p^j & \text{for } i=4j, \\ Z_p^{j+1} & \text{for } i=4j+1, 4j+2, 4j+3, \end{cases}$$

generated by A, and

$$H^{4n+2-i}(L^n(p)^*; \underline{Z}) = \begin{cases} Z_p^j & \text{for } i=4j, 4j+1, 4j+2, \\ Z_p^{j+1} & \text{for } i=4j+3, \end{cases}$$

generated by B.

As for the cohomology groups $H^i(L^n(p)^*; Z_3)$ and $H^i(L^n(p)^*; \underline{Z}_3)$, it follows that

LEMMA 2.11. *The following relations hold.*
(1) *If $p \neq 3$, then*

$$H^t(L^n(p)^*; Z_3) = 0, \qquad H^t(L^n(p)^*; \underline{Z}_3) = 0 \qquad \text{for } t > 2n+1.$$

(2) *If $p=3$, then*

$H^{4n+1}(L^n(3)^*; Z_3) = Z_3$ *generated by* $yx^n \times x^n + x^n \times yx^n$,

$H^{4n}(L^n(3)^*; Z_3) = Z_3 + Z_3$ *generated by* $\{yx^n \times yx^{n-1} - yx^{n-1} \times yx^n, x^n \times x^n\}$,

$H^{4n+1}(L^n(3)^*; \underline{Z}_3) = 0, \qquad H^{4n}(L^n(3)^*; \underline{Z}_3) = 0,$

$H^{4n-1}(L^n(3)^*; \underline{Z}_3) = Z_3$ *generated by* $x^n \times yx^{n-1} - yx^{n-1} \times x^n = yx^n \times x^{n-1} - x^{n-1} \times yx^n$.

§3. Proof of Theorem A

It is known that $L^n(p)$ is embedded in R^m for $m \geq 3(2n+1)/2$, (cf. e.g., [13, Theorem 1.1]). We prove (4) and (5) for $p=3$ only. The others are obtained easily by the same way.

PROOF OF (4) FOR $p=3$. The group $[L^n(3) \subset R^{4n-2}] = [L^n(3)^*, RP^{4n-3}; g]$ in the introduction is clearly isomorphic to $[L^n(3)^*, (RP^{4n-3})^2; f]$, where f: $L^n(3)^* \to RP^{4n-3}$ is a fixed lifting of g: $L^n(3)^* \to RP^\infty$. Therefore

$$[L^n(3) \subset R^{4n-2}] \approx [L^n(3)^*, E_4; f]$$

by the dimensional reason. By Lemma 2.4, the homotopy exact sequence of fibrations p_i $(i=2, 3, 4)$ in §1 induces isomorphisms

$$[L^n(3)^*, E_4; f] \xrightarrow[\approx]{p_{4\sharp}} [L^n(3)^*, E_3; f] \xrightarrow[\approx]{p_{3\sharp}} [L^n(3)^*, E_2; f]$$

and an exact sequence

$$H^{4n-4}(L^n(3)^*; Z) \xrightarrow{\Theta^{4n-3}} H^{4n}(L^n(3)^*; Z_3) \xrightarrow{i_\sharp} [L^n(3)^*, E_2; f]$$
$$\xrightarrow{p_{2\sharp}} H^{4n-3}(L^n(3)^*; Z) \xrightarrow{\Theta^{4n-2}} H^{4n+1}(L^n(3)^*; Z_3).$$

Here $\Theta^i = \mathscr{P}_3^1 \rho_3$ for $i = 4n-2, 4n-3$ by Proposition 1.1.

To determine Θ^i, consider the commutative diagram

$$\begin{array}{ccc}
H^i(L^n(3)^*; Z) & \xrightarrow{\Theta^{i+1} = \mathscr{P}_3^1 \rho_3} & H^{i+4}(L^n(3)^*; Z_3) \\
\approx \downarrow \pi^* & & \approx \downarrow \pi^* \\
H^i(L^n(3) \times L^n(3) - \Delta; Z)^{t^*} & \xrightarrow{\mathscr{P}_3^1 \rho_3} & H^{i+4}(L^n(3) \times L^n(3) - \Delta; Z_3)^{t^*} \\
\approx \uparrow i^* & & \uparrow i^* \\
(H^i(L^n(3) \times L^n(3); Z)/\mathrm{Ker}\, i^*)^{t^*} & \xrightarrow{\mathscr{P}_3^1 \rho_3} & (H^{i+4}(L^n(3) \times L^n(3); Z_3)/\mathrm{Ker}\, i^*)^{t^*}.
\end{array}$$

In this diagram, π^*'s are isomorphisms by Lemma 2.6 and i^* in the left hand side is an isomorphism by Lemma 2.7 and (2.8). By the use of this diagram, Proposition 2.10 and Lemma 2.11, a simple calculation yields that

$$\mathrm{Ker}\, \Theta^{4n-2} = \begin{cases} Z_3 + Z_3 \text{ generated by } \{\delta_3(yx^n \times yx^{n-3} - yx^{n-3} \times yx^n), \\ \qquad \delta_3(yx^{n-1} \times yx^{n-2} - yx^{n-2} \times yx^{n-1})\}, \quad n \equiv 2(3), \\ Z_3 \text{ generated by } \delta_3(yx^n \times yx^{n-3} - yx^{n-3} \times yx^n) + \\ \qquad \delta_3(yx^{n-1} \times yx^{n-2} - yx^{n-2} \times yx^{n-1}), \quad n \not\equiv 2(3); \end{cases}$$

$$\operatorname{Coker}\Theta^{4n-3}=\begin{cases} Z_3+Z_3 \textit{ generated by } \{yx^n\times yx^{n-1}-yx^{n-1}\times yx^n, \\ \qquad\qquad x^n\times x^n\}, & n\equiv 2(3), \\ Z_3 \textit{ generated by } yx^n\times yx^{n-1}-yx^{n-1}\times yx^n, \\ & n\not\equiv 2(3). \end{cases}$$

This result and the above exact sequence give rise to the exact sequences

$$0 \longrightarrow Z_3+Z_3 \xrightarrow{i_\sharp} [L^n(3)^*, E_2; f] \xrightarrow{p_{2\sharp}} Z_3+Z_3 \longrightarrow 0, \qquad n\equiv 2(3),$$

$$0 \longrightarrow Z_3 \xrightarrow{i_\sharp} [L^n(3)^*, E_2; f] \xrightarrow{p_{2\sharp}} Z_3 \longrightarrow 0, \qquad n\not\equiv 2(3).$$

To consider the group extensions of these exact sequences, let

$$\Phi(3, 1)\colon \operatorname{Ker}\Theta^{4n-2} \longrightarrow \operatorname{Coker}\Theta^{4n-3}$$

be the homomorphism defined by

$$\Phi(3, 1)(a) = b, \qquad i_\sharp(b) = 3p_{2\sharp}^{-1}(a).$$

LEMMA 3.1. $\qquad \Phi(3, 1) = \mathscr{P}_3^1\delta_3^{-1}.$

PROOF. Let $p_2'\colon E_2'\to K(Z, 4n-3)$ be the principal fibration with classifying map $\mathscr{P}_3^1\rho_3\colon K(Z, 4n-3)\to K(Z_3, 4n+1)$ and consider the commutative diagram of fibrations in the category $\mathscr{X}_{RP^{4n-3}}$ (see [11, 1]).

$$\begin{array}{ccc}
RP^{4n-3}\times K(Z_3, 4n) & \subset & RP^{4n-3}\times K(Z_2, 4n-2)\times K(Z_2, 4n)\times K(Z_3, 4n) \\
\downarrow & & \downarrow \\
RP^{4n-3}\times E_2' & \subset & E_2 \\
\downarrow{\scriptstyle 1\times p_2'} & & \downarrow{\scriptstyle p_2} \\
RP^{4n-3}\times K(Z, 4n-3) & = & E_1 \\
\downarrow{\scriptstyle 1\times\mathscr{P}_3^1\rho_3} & & \downarrow{\scriptstyle \Theta^{4n-2}} \\
RP^{4n-3}\times K(Z_3, 4n+1) & \subset & RP^{4n-3}\times K(Z_2, 4n-1)\times K(Z_2, 4n+1) \\
 & & \times K(Z_3, 4n+1).
\end{array}$$

Since $H^i(L^n(3)^*; Z_2)=0$ for $i>2n+1$ by Lemma 2.4, the homotopy exact sequences and the five lemma yield a commutative diagram of exact sequences

$$
\begin{array}{ccccc}
 & & H^{4n-4}(L^n(3)^*;\, Z) & & \\
 & & \downarrow & & \\
 & & H^{4n}(L^n(3)^*;\, Z_3) & & \\
 & \swarrow & \downarrow & \searrow & \\
[L^n(3)^*,\, E_2'] & \approx & [L^n(3)^*,\, RP^{4n-3}\times E_2';\, f] & \approx & [L^n(3)^*;\, E_2;\, f] \\
 & \searrow & \downarrow & \swarrow & \\
 & & H^{4n-3}(L^n(3)^*;\, Z) & & \\
 & & \downarrow & & \\
 & & H^{4n+1}(L^n(3)^*;\, Z_3). & &
\end{array}
$$

Considering the left exact sequence, we can easily verify that $\Phi(3, 1)$ coincides with $\Phi(3, 1)$ in [10, 1]. By [10, Corollary 3.7. Case II], we have $\Phi(3, 1)=\mathscr{P}^1_3\delta_3^{-1}$.

This lemma shows the relations

$$
\begin{aligned}
&\Phi(3, 1)(\delta_3(yx^n \times yx^{n-3} - yx^{n-3} \times yx^n)) \\
&\qquad = (n-3)(yx^n \times yx^{n-1} - yx^{n-1} \times yx^n), \\
&\Phi(3, 1)(\delta_3(yx^{n-1} \times yx^{n-2} - yx^{n-2} \times yx^{n-1})) \\
&\qquad = (n-2)(yx^{n-1} \times yx^n - yx^n \times yx^{n-1}).
\end{aligned}
$$

These relations imply that

$$
[L^n(3) \subset R^{4n-2}] = [L^n(3)^*,\, E_2;\, f] = \begin{cases} Z_3 + Z_3 + Z_9 & \text{for} \quad n \equiv 2(3), \\ Z_9 & \text{for} \quad n \not\equiv 2(3). \end{cases}
$$

Proof of (5) for $p=3$. By the same way as in the proof of (4) for $p=3$, there are an isomorphism

$$
[L^n(3) \subset R^{4n-3}] = [L^n(3)^*,\, E_2;\, f],
$$

and an exact sequence

$$
\begin{aligned}
&H^{4n-5}(L^n(3)^*;\, \underline{Z}) \xrightarrow{\Theta^{4n-4}=\mathscr{P}^1_3\rho_3} H^{4n-1}(L^n(3)^*;\, \underline{Z}_3) \longrightarrow \\
&\qquad [L^n(3)^*,\, E_2;\, f] \longrightarrow H^{4n-4}(L^n(3)^*;\, \underline{Z}) \xrightarrow{\Theta^{4n-3}} H^{4n}(L^n(3)^*;\, \underline{Z}_3).
\end{aligned}
$$

Since $H^{4n-4}(L^n(3)^*;\, \underline{Z})=Z_3$ and $H^{4n}(L^n(3)^*;\, \underline{Z}_3)=0$ by Proposition 2.10 and Lemma 2.11, it is sufficient to show that $\Theta^{4n-4}=\mathscr{P}^1_3\rho_3$ is an epimorphism. Consider the diagram

$$\begin{array}{ccc}
H^{4n-5}(L^n(3)^*; \underline{Z}) & \xrightarrow{\mathscr{P}_3^1\rho_3} & H^{4n-1}(L^n(3)^*; Z_3) \\
\approx\downarrow\pi^* & & \approx\downarrow\pi^* \\
H^{4n-5}(L^n(3)\times L^n(3)-\Delta; Z)^{-t^*} & \xrightarrow{\mathscr{P}_3^1\rho_3} & H^{4n-1}(L^n(3)\times L^n(3)-\Delta; Z_3)^{-t^*} \\
\approx\uparrow i^* & & \uparrow i^* \\
(H^{4n-5}(L^n(3)\times L^n(3); Z)/\mathrm{Ker}\, i^*)^{-t^*} & \xrightarrow{\mathscr{P}_3^1\rho_3} & (H^{4n-1}(L^n(3)\times L^n(3); Z_3)/\mathrm{Ker}\, i^*)^{-t^*}.
\end{array}$$

Here π^*'s are isomorphisms by Lemma 2.6 and i^* in the left hand side is an isomorphism by Lemma 2.7, and the last two $\mathscr{P}_3^1$'s are the ordinary reduced power operations mod 3 and the first $\mathscr{P}_3^1$ is the twisted one (see Proposition 1.1). By using Proposition 2.10, there are relations

$$\mathscr{P}_3^1\rho_3(\delta_3(yx^{n-1}\times yx^{n-3}+yx^{n-3}\times yx^{n-1})) = (2n-5)(x^n\times yx^{n-1}-yx^{n-1}\times x^n),$$

$$\mathscr{P}_3^1\rho_3(\delta_3(yx^{n-2}\times yx^{n-2})) = (2-n)(x^n\times yx^{n-1}-yx^{n-1}\times x^n).$$

If $n-2\equiv 0(3)$, then $2n-5\not\equiv 0(3)$. Hence Θ^{4n-4} is an epimorphism by Lemma 2.11.

§4. The cohomology of $(RP^n)^*$ and $(CP^n)^*$

This section is devoted to determine some cohomology groups of $(RP^n)^*$ and $(CP^n)^*$.

Let F denote the real field R or the complex field C and let d be 1 or 2 according as $F=R$ or C, and let $G_{n+1,2}(F)$ denote the Grassmann manifold of 2-planes in F^{n+1}. The cohomology ring of $G_{n+1,2}(F)$ is well-known and is given as follows:

(4.1)
$$H^*(G_{n+1,2}(F); G) = G[x, y]/(a_n, a_{n+1}) \qquad (G=Z_2 \text{ if } F=R, \ =Z \text{ if } F=C),$$

where $\deg x=d$, $\deg y=2d$ *and* $a_r=\sum_i\binom{r-i}{i}x^{r-2i}y^i$ $(r=n, n+1)$. *Moreover, there are relations*

$$x^{2i}y^{n-1-i}=0 \quad \text{if} \quad i\neq 2^t-1 \quad \text{for some} \quad t, \text{ (cf. [5, Corollary 4.1])}$$

$$x^{2^{r+1}-1}=0,\ x^{2^{r+1}-2}y^s\neq 0 \quad \text{for} \quad n=2^r+s\ (0\le s<2^r).$$

The mod 2 *cohomology ring of* $G_{n+1,2}(C)$ *is given by*

$$H^*(G_{n+1,2}(C); Z_2) = Z_2[x, y]/(a_n, a_{n+1}),$$

where x, y and a_r $(r=n, n+1)$ are the mod 2 *reduction of the same symbols in the integral cohomology. Further, there is a relation*

$$Sq^d x = xy.$$

The last relation for $F=R$ and the induction lead to the following lemma. Details will be omitted.

LEMMA 4.2. *There are the following relations in $H^*(G_{n+1,2}(R); Z_2)$.*

(1) $Sq^2 y^t = t y^{t+1} + \binom{t}{2} x^2 y^t$.

(2) $Sq^3 y^t = \alpha_t x^3 y^t$,

$$\alpha_t = \sum_{0<i<t} \binom{i}{2} \equiv \begin{cases} 0(2) & \text{for } t \not\equiv 3(4), \\ 1(2) & \text{for } t \equiv 3(4). \end{cases}$$

(3) $Sq^4 y^t = \binom{t}{2} y^{t+2} + \alpha_t x^2 y^{t+1} + \beta_t x^4 y^t$,

$$\beta_t = \sum_{0<i<t} \alpha_i \equiv \begin{cases} 0(2) & \text{for } t \equiv l(8), \ 0 \le l \le 3, \\ 1(2) & \text{for } t \equiv l(8), \ 4 \le l \le 7. \end{cases}$$

Case I. $(RP^n)^*$.

The mod 2 cohomology ring of $(RP^n)^*$ is investigated by S. Feder [4], [5] and D. Handel [8] and is given as follows:

(4.3) *$(RP^n)^*$ has the homotopy type of a $(2n-1)$-dimensional closed manifold and $H^*((RP^n)^*; Z_2)$ has $\{1, v\}$ as a basis of an $H^*(G_{n+1,2}(R); Z_2)$-module, where v is the first Stiefel-Whitney class of the double covering $RP^n \times RP^n - \Delta \to (RP^n)^*$ and the ring structure is given by the relation*

$$v^2 = vx.$$

The group structure of $H^t((RP^n)^*; Z_2)$ and its basis for $2n-4 \le t \le 2n-1$ are determined by the Poincaré duality and are given in [19, (6.3)] and [19, (8.3)]. By the same way, we have

(4.4) *Let $n=2^r+s$, $2<s<2^r$. Then the* mod 2 *cohomology groups $H^t((RP^n)^*; Z_2)$ for $2n-8 \le t \le 2n-5$ are given in the table below.*

t	$H^t((RP^n)^*;Z_2)$	basis
$2n-5$	Z_2^5	$x^{2^{r+1}-5+2i}y^{s-i} (i=0,1)$, $vx^{2^{r+1}-6+2i}y^{s-i} (0 \le i \le 2)$
$2n-6$	Z_2^6	$x^{2^{r+1}-6+2i}y^{s-i} (0 \le i \le 2)$, $vx^{2^{r+1}-7+2i}y^{s-i} (0 \le i \le 2)$
$2n-7$	Z_2^7	$x^{2^{r+1}-7+2i}y^{s-i} (0 \le i \le 2)$, $vx^{2^{r+1}-8+2i}y^{s-i} (0 \le i \le 3)$
$2n-8$	Z_2^8	$x^{2^{r+1}-8+2i}y^{s-i} (0 \le i \le 3)$, $vx^{2^{r+1}-9+2i}y^{s-i} (0 \le i \le 3)$

Now, $H^*((RP^n)^*; \underline{Z})$ and $H^*((RP^n)^*; \underline{Z}_3)$ are the cohomology with coefficients in the local system on $(RP^n)^*$ determined by $v \in H^1((RP^n)^*; Z_2)$.

(4.5) ([9, p. 481]) *The groups* $H^t((RP^n)^*; Z)$ *and* $H^t((RP^n)^*; \underline{Z})$ *are* 2-*primary groups for* $n<t<2n-1$.

Consider the Bockstein exact sequences (2.1) for $q=2$ and for $(RP^n)^*$. Then there are relations

$$\rho_2\delta_2 = Sq^1, \qquad \rho_2\tilde{\delta}_2 = Sq^1 + v. \tag{4.6}$$

By (4.4–6), we can easily verify the following results.

LEMMA 4.7. *Let* $n \equiv 0(2)$, $n=2^r+s$ $(3 \le s < 2^r)$. *Then we have*

$H^{2n-5}((RP^n)^*; Z) = Z_2^2$ *generated by* $\{\delta_2(vx^{2^{r+1}-5}y^{s-1}),$
$\delta_2(x^{2^{r+1}-4}y^{s-1})\}$,

$H^{2n-6}((RP^n)^*; Z) = Z_2^4$ *generated by* $\{\delta_2(vx^{2^{r+1}-8}y^s), \delta_2(x^{2^{r+1}-7}y^s),$
$\delta_2(vx^{2^{r+1}-4}y^{s-2}), \delta_2(x^{2^{r+1}-3}y^{s-2})\}$,

$\rho_2 H^{2n-7}((RP^n)^*; Z) = Z_2^3$ *generated by* $\{vx^{2^{r+1}-6}y^{s-1}, x^{2^{r+1}-5}y^{s-1},$
$vx^{2^{r+1}-2}y^{s-3}\}$;

$H^{2n-4}((RP^n)^*; \underline{Z}) = Z_2^2$ *generated by* $\{\tilde{\delta}_2(x^{2^{r+1}-5}y^s),$
$\tilde{\delta}_2(x^{2^{r+1}-3}y^{s-1})\}$,

$H^{2n-5}((RP^n)^*; \underline{Z}) = Z_2^3$ *generated by* $\{\tilde{\delta}_2(x^{2^{r+1}-6}y^s),$
$\tilde{\delta}_2(x^{2^{r+1}-4}y^{s-1}), \tilde{\delta}_2(x^{2^{r+1}-2}y^{s-2})\}$,

$H^{2n-6}((RP^n)^*; \underline{Z}) = Z_2^3$ *generated by* $\{\tilde{\delta}_2(x^{2^{r+1}-7}y^s),$
$\tilde{\delta}_2(x^{2^{r+1}-5}y^{s-1}), \tilde{\delta}_2(x^{2^{r+1}-3}y^{s-2})\}$,

$H^{2n-7}((RP^n)^*; \underline{Z}) = Z_2^4$ *generated by* $\{\tilde{\delta}_2(x^{2^{r+1}-8}y^s), \tilde{\delta}_2(x^{2^{r+1}-6}y^{s-1}),$
$\tilde{\delta}_2(x^{2^{r+1}-4}y^{s-2}), \tilde{\delta}_2(x^{2^{r+1}-2}y^{s-3})\}$;

$H^{2n-1}((RP^n)^*; Z_3) = Z_3, \qquad H^{2n-1}((RP^n)^*; \underline{Z}_3) = 0.$

Case II. $(CP^n)^*$.

The integral and the mod 2 cohomology of $(CP)^{n*}$ are investigated by S. Feder [5] and the author [18], and are given as follows:

(4.8) *$(CP^n)^*$ has the homotopy type of an unorientable $(4n-2)$-dimensional closed manifold and $H^*((CP^n)^*; Z_2)$ has $\{1, v, v^2\}$ as basis of an $H^*(G_{n+1,2}(C); Z_2)$-module and $H^*((CP^n)^*; Z)$ has $\{1, u\}$ as generators of an $H^*(G_{n+1,2}(C); Z)$-module, where v is the first Stiefel-Whitney class of the double covering $CP^n \times CP^n - \Delta \to (CP^n)^*$ and $u = \delta_2 v$. The ring structures are given by the relations*

$$v^3 = vx, \qquad u^2 = ux.$$

Then the integral and the mod 2 cohomology groups of $(CP^n)^*$ are given by the following.

(4.9) *Let $n = 2^r + s$ $(0 < s < 2^r)$. Then we have*

t	$H^t((CP^n)^*; Z_2)$	basis
$4n-2$	Z_2	$v^2 x^{2^{r+1}-2} y^s$
$4n-3$	Z_2	$v x^{2^{r+1}-2} y^s$
$4n-4$	$Z_2 + Z_2$	$x^{2^{r+1}-2} y^s$, $v^2 x^{2^{r+1}-3} y^s$
$4n-5$	Z_2	$v x^{2^{r+1}-3} y^s$
$4n-6$	$Z_2 + Z_2 + Z_2$	$x^{2^{r+1}-3} y^s$, $v^2 x^{2^{r+1}-4} y^s$, $v^2 x^{2^{r+1}-2} y^{s-1}$
$4n-7$	$Z_2 + Z_2$	$v x^{2^{r+1}-4} y^s$, $v x^{2^{r+1}-2} y^{s-1}$

$H^{4n-6}((CP^n)^*; Z) = Z + Z_2 + Z_2$ generated by $\{x^{2^{r+1}-3} y^s,$

$u x^{2^{r+1}-4} y^s,\ u x^{2^{r+1}-2} y^{s-1}\}$,

$H^i((CP^n)^*; Z) = 0$ for odd i.

Using the Poincaré duality $H^{4n-2-i}((CP^n)^*; \underline{Z}) = H_i((CP^n)^*; Z)$ and the Bockstein exact sequence (2.1), we can show the following:

(4.10) *Let $n = 2^r + s$ $(0 < s < 2^r)$.*

$H^{4n-4}((CP^n)^*; \underline{Z}) = Z$ generated by a with

$$\rho_2(a) = v^2 x^{2^{r+1}-3} y^s + x^{2^{r+1}-2} y^s,$$

$H^{4n-5}((CP^n)^*; \underline{Z}) = Z_2$ generated by $\rho_2^{-1}(v x^{2^{r+1}-3} y^s)$,

$H^{4n-6}((CP^n)^*; \underline{Z}) = Z + Z$ generated by $\{b, b'\}$ with

$$\rho_2(b) = v^2 x^{2^{r+1}-4} y^s + x^{2^{r+1}-3} y^s,$$

$$\rho_2(b') = v^2 x^{2^{r+1}-2} y^{s-1},$$

$H^{4n-7}((CP^n)^*;\ \underline{Z}) = Z_2 + Z_2$ *generated by* $\{\rho_2^{-1}(vx^{2^{r+1}-4}y^s),$

$\rho_2^{-1}(vx^{2^{r+1}-2}y^{s-1})\};$

$H^{4n-2}((CP^n)^*;\ \underline{Z}_3) = Z_3, \qquad H^{4n-3}((CP^n)^*;\ \underline{Z}_3) = 0;$

$H^{4n-2}((CP^n)^*;\ Z_3) = 0, \qquad H^{4n-3}((CP^n)^*;\ Z_3) = 0.$

§5. Proofs of Theorems B and C

PROOF OF THEOREM B. We prove (1) only. The others are similar and will be omitted. By applying Proposition 1.1 for $(RP^n)^*$ and $2n-4$ in place of X and n, respectively, there follows a decreasing filtration

$$[RP^n \subset R^{2n-3}] = F_0 \supset F_1 \supset F_2 \supset F_3 \supset 0$$

such that

$$F_0/F_1 = \operatorname{Ker}\chi^{2n-3}, \qquad F_1/F_2 = \operatorname{Ker}\Psi^{2n-3},$$

$$F_2/F_3 = \operatorname{Coker}\Phi^{2n-4}, \qquad F_3 = \operatorname{Coker}\chi^{2n-4},$$

where Φ^i, Ψ^i and χ^i are the secondary and the tertiary operations defined by the homomorphisms

$$\Theta^i\colon H^{i-1}((RP^n)^*;\ \underline{Z}) \longrightarrow$$
$$H^{i+1}((RP^n)^*;\ Z_2) \times H^{i+3}((RP^n)^*;\ Z_2) \times H^{i+3}((RP^n)^*;\ \underline{Z}_3),$$

$$\Theta^i(a) = \begin{cases} (Sq^2\rho_2 a,\ Sq^4\rho_2 a + v^4\rho_2 a,\ \mathscr{P}_3^1\rho_3 a), & n \equiv 0(4), \\ (Sq^2\rho_2 a,\ Sq^4\rho_2 a,\ \mathscr{P}_3^1\rho_3 a), & n \equiv 2(4); \end{cases}$$

$$\Gamma^i\colon H^i((RP^n)^*;\ Z_2) \times H^{i+2}((RP^n)^*;\ Z_2) \times H^{i+2}((RP^n)^*;\ \underline{Z}_3) \longrightarrow$$
$$H^{i+2}((RP^n)^*;\ Z_2) \times H^{i+3}((RP^n)^*;\ Z_2),$$

$$\Gamma^i(a, b, c) = ((Sq^2 + vSq^1 + v^2)a,\ (Sq^2Sq^1 + v^2Sq^1)a + (Sq^1 + v)b);$$

$$\Delta^i\colon H^{i+1}((RP^n)^*;\ Z_2) \times H^{i+2}((RP^n)^*;\ Z_2) \longrightarrow H^{i+3}((RP^n)^*;\ Z_2),$$

$$\Delta^i(a, b) = Sq^2 a + v^2 a + Sq^1 b + vb.$$

Using the results of §4, we can easily verify that

$$\operatorname{Ker}\Theta^{2n-3} = \begin{cases} Z_2, & n \equiv 0(4), \\ 0, & n \equiv 2(4), \end{cases}$$

$$\operatorname{Im}\Gamma^{2n-3} = \operatorname{Ker}\Delta^{2n-2}, \qquad \operatorname{Im}\Gamma^{2n-4} = \operatorname{Ker}\Delta^{2n-3},$$

$$\operatorname{Ker}\Gamma^{2n-3} = Z_2 + Z_2 + Z_2,$$

$$\operatorname{Coker}\Delta^{2n-3} = 0, \qquad \operatorname{Im}\Theta^{2n-4} = \begin{cases} Z_2 + Z_2 + Z_2, & n \equiv 0(4), \\ Z_2 + Z_2, & n \equiv 2(8), \\ Z_2, & n \equiv 6(8), \end{cases}$$

$$\operatorname{Coker}\Delta^{2n-4} = 0,$$

Hence it follows that

$$\operatorname{Ker}\Phi^{2n-3} = \operatorname{Ker}\Theta^{2n-3}, \qquad \operatorname{Ker}\chi^{2n-3} = \operatorname{Ker}\Phi^{2n-3},$$

$$\operatorname{Ker}\Psi^{2n-3} = \operatorname{Ker}\Gamma^{2n-3}/\operatorname{Im}\Theta^{2n-4} = \begin{cases} 0, & n \equiv 0(4), \\ Z_2, & n \equiv 2(8), \\ Z_2 + Z_2, & n \equiv 6(8), \end{cases}$$

$$\operatorname{Coker}\Phi^{2n-4} = 0, \qquad \operatorname{Coker}\chi^{2n-4} = 0.$$

This implies that

$$[RP^n \subset R^{2n-3}] = \begin{cases} Z_2, & n \not\equiv 6(8), \\ Z_2 + Z_2, & n \equiv 6(8). \end{cases}$$

REMARK OF THEOREM B. In (3) for $n \equiv 2(4)$, the secondary and the tertiary operations cannot be calculated. Therefore $[RP^n \subset R^{2n-5}]$ for $n \equiv 2(4)$ is not determined and so is $[RP^n \subset R^{2n-i}]$ ($i=3, 4, 5$) for $n \equiv 1(2)$ by the same reason.

PROOF OF THEOREM C. We can prove (1) only. (2) and (3) are obtained by the same way. By Proposition 1.1, there is a decreasing filtration

$$[CP^n \subset R^{4n-3}] = F_0 \supset F_1 \supset F_2 \supset F_3 \supset 0$$

such that

$$F_0/F_1 = \operatorname{Ker}\chi^{4n-3}, \qquad F_1/F_2 = \operatorname{Ker}\Psi^{4n-3},$$

$$F_2/F_3 = \operatorname{Coker}\Phi^{4n-4}, \qquad F_3 = \operatorname{Coker}\chi^{4n-4},$$

where Φ^i, Ψ^i and χ^i are the secondary and the tertiary operations defined by the homomorphisms

$$\Theta^i\colon H^{i-1}((CP^n)^*; \underline{Z}) \longrightarrow$$
$$H^{i+1}((CP^n)^*; Z_2) \times H^{i+3}((CP^n)^*; Z_2) \times H^{i+3}((CP^n)^*; \underline{Z}_3),$$

$$\Theta^i(a) = \begin{cases} (Sq^2\rho_2 a,\ Sq^4\rho_2 a + v^4\rho_2 a,\ \mathscr{P}_3^1\rho_3 a), & n \equiv 0(2), \\ (Sq^2\rho_2 a,\ Sq^4\rho_2 a,\ \mathscr{P}_3^1\rho_3 a), & n \equiv 1(2); \end{cases}$$

$$\Gamma^i: H^i((CP^n)^*; Z_2) \times H^{i+2}((CP^n)^*; Z_2) \times H^{i+2}((CP^n)^*; \underline{Z}_3) \longrightarrow H^{i+2}((CP^n)^*; Z_2) \times H^{i+3}((CP^u)^*; Z_2),$$

$$\Gamma^i(a, b, c) = ((Sq^2 + vSq^1 + v^2)a, (Sq^2Sq^1 + v^2Sq^1)a + (Sq^1 + v)b);$$

$$\Delta^i: H^{i+1}((CP^n)^*; Z_2) \times H^{i+2}((CP^n)^*; Z_2) \longrightarrow H^{i+3}((CP^n)^*; Z_2),$$

$$\Delta^i(a, b) = Sq^2a + v^2a + Sq^1b + vb.$$

By (4.1) and (4.8–10), there are the relations.

$$\operatorname{Ker} \Theta^{4n-3} = Z, \qquad \operatorname{Ker} \Theta^{4n-4} = \begin{cases} 0, & n \equiv 0(2), \\ Z_2, & n \equiv 1(2), \end{cases}$$

$$\operatorname{Im} \Theta^{4n-4} = \begin{cases} Z_2, & n \equiv 0(2), \\ 0, & n \equiv 1(2), \end{cases} \qquad \operatorname{Ker} \Gamma^{4n-3} = Z_2,$$

$$\operatorname{Im} \Gamma^{4n-3} = \operatorname{Ker} \Delta^{4n-2} = 0, \qquad \operatorname{Im} \Gamma^{4n-3} = \operatorname{Ker} \Delta^{4n-3},$$

$$\operatorname{Coker} \Delta^{4n-4} = 0, \qquad \operatorname{Coker} \Delta^{4n-4} = 0.$$

Hence it follows that

$$\operatorname{Ker} \Phi^{4n-3} = \operatorname{Ker} \Theta^{4n-3} = Z, \qquad \operatorname{Ker} \chi^{4n-3} = \operatorname{Ker} \Phi^{4n-3} = Z,$$

$$\operatorname{Ker} \Psi^{4n-3} = \operatorname{Ker} \Gamma^{4n-3}/\operatorname{Im} \Theta^{4n-4} = \begin{cases} 0, & n \equiv 0(2), \\ Z_2, & n \equiv 1(2), \end{cases}$$

$$\operatorname{Coker} \Phi^{4n-3} = 0, \qquad \operatorname{Coker} \chi^{4n-4} = 0.$$

Therefore, if $n \equiv 0(2)$, then $[CP^n \subset R^{4n-3}] = F_0 = Z$, and if $n \equiv 1(2)$, then $0 \to Z_2 \to F_0 \to Z \to 0$ is a short exact sequence. This completes the proof.

REMARK OF THEOREM C. As for $[CP^n \subset R^{4n-i}]$ ($i=4, 5$) for $n \equiv 1(2)$, the following are verified.

(2)′ $$\#[CP^n \subset R^{4n-4}] = \begin{cases} 2 \text{ or } 4, & n \equiv 1(4), \\ 4 \text{ or } 8, & n \equiv 3(4); \end{cases}$$

(3)′ $[CP^n \subset R^{4n-5}] = Z + Z + G,$

$$\#G = \begin{cases} 1 \text{ or } 2, & n \equiv 1(4), \\ 2 \text{ or } 4, & n \equiv 3(4). \end{cases}$$

References

[1] D. R. Bausum, *Embeddings and immersions of manifolds in Euclidean space*, Trans. Amer. Math. Soc., **213** (1975), 263–303.

[2] J. C. Becker, *Cohomology and the classification of liftings*, Trans. Amer. Math. Soc., **133** (1968), 447–475.

[3] A. Borel, *Sur la cohomologie des espace fibrés principaux et des espace homogènes de groupes de Lei compacts*, Ann. of Math., **57** (1953), 115–207.

[4] S. Feder, *The reduced symmetric product of a projective space and the embedding problem*, Bol. Soc. Mat. Mexicana, **12** (1967), 76–80.

[5] ———, *The reduced symmetric product of projective spaces and the generalized Whitney theorem*, Illinois J. Math., **16** (1972), 323–329.

[6] S. Gitler, *Cohomology operations with local coefficients*, Amer. J. Math., **85** (1963), 156–188.

[7] A. Haefliger, *Plongements différentiables dans le domaine stable*, Comment. Math. Helv., **37** (1962), 155–176.

[8] D. Handel, *An embedding theorem for real projective spaces*, Topology, **7** (1968), 125–130.

[9] L. L. Larmore and R. D. Rigdon, *Enumerating immersions and embeddings of projective spaces*, Pacific J. Math., **64** (1976), 471–492.

[10] L. L. Larmore and E. Thomas, *Group extensions and principal fibrations*, Math. Scand., **30** (1972), 227–248.

[11] ——— and ———, *Group extensions and the twisted cohomology theories*, Illinois J. Math., **17** (1973), 397–410.

[12] J. C. McClendon, *Obstruction theory in fiber spaces*, Math. Z., **120** (1971), 1–17.

[13] E. Rees, *Embedding odd torsion manifolds*, Bull. London Math. Soc., **3** (1971), 356–362.

[14] ———, *Embeddings of real projective spaces*, Topology, **10** (1971), 309–312.

[15] E. H. Spanier, Algebraic Topology, McGraw-Hill, 1966.

[16] B. Steer, *On the embedding of projective spaces in Euclidean space*, Proc. London Math. Soc., **21** (1970), 489–501.

[17] E. Thomas, *Embedding Manifolds in Euclidean space*, Osaka J. Math., **13** (1976), 163–186.

[18] T. Yasui, *The reduced symmetric product of a complex projective space and the embedding problem*, Hiroshima Math. J., **1** (1971), 27–40; **2** (1972), 233–237.

[19] ———, *Note on the enumeration of embeddings of real projective spaces*, Hiroshima Math. J., **3** (1973), 409-418; **6**(1976), 221–225.

[20] ———, *The enumeration of liftings in fibrations and the embedding problem* I, Hiroshima Math. J., **6** (1976), 227–255.

Department of Mathematics,
Faculty of Education,
Yamagata University