

Immersion of real projective spaces into complex projective spaces

Dedicated to Professor Masahiro Sugawara on his 60th birthday

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§1. Introduction

Given two differentiable manifolds M, N and a continuous map $f: M \rightarrow N$, we denote by $I[M, N]_f$ the set of regular homotopy classes of immersions of M into N , which are homotopic to f , and by $I[M, N]$ the set of all regular homotopy classes of immersions of M into N .

As for the set $I[M, N]$, some results have so far been obtained when N is not Euclidean space. For example, for the existence of immersions of $P^n(Q)$ into $P^m(C)$ or $P^m(R)$, of $P^n(C)$ into $P^m(C)$, of $P^n(R)$ into $P^m(R)$, and of $L^n(p)$ into $L^m(p)$, see [1], [4] and [15], [8], and [5], respectively, and for the classification of immersions of $P^n(R)$ into $P^m(R)$, see [7]–[9], where $P^k(F)$ is the F -projective space of F -dimension k and $L^k(p)$ is the lens space mod p . But the above results are smaller in number than those when N is Euclidean space.

In this article we shall study the set $I[P^n(R), P^m(C)]$ and $I[P^n(R), P^m(C)]_f$ for any map $f: P^n(R) \rightarrow P^m(C)$ ($n \leq 2m - 1$).

Here we note the fact that $[P^n(R), P^m(C)] = Z_2$ if $n \leq 2m$ (see (2.5) below).

Let $i: P^n(R) \rightarrow P^n(C) \subset P^m(C)$ ($n \leq m$) be the natural embedding defined by regarding real numbers as complex numbers and let $c: P^n(R) \rightarrow P^m(C)$ be a constant map. Then we shall prove the following theorems:

THEOREM A. *Assume that $n \geq 2$. Then*

- (i) *for $n \leq m$, the natural embedding i is not null-homotopic,*
- (ii) *for $n > m$, any immersion of $P^n(R)$ into $P^m(C)$, if any, is always null-homotopic.*

THEOREM B. *Assume that $n > 2$. Then*

- (i) *both for $f=i$ and $f=c$,*

$$I[P^n(R), P^m(C)]_f = \begin{cases} Z & n \equiv 0 \pmod{2}, \\ Z_2 & n \equiv 1 \pmod{2}; \end{cases}$$

- (ii) *if $m < n < 2m - 1$, then*

$$I[P^n(R), P^m(C)] = I[P^n(R), P^m(C)]_c = I[P^n(R), R^{2m}].$$

REMARK. (i) According to Whitney [17], for $n \leq m$ there exists a null-homotopic immersion of $P^n(R)$ into $P^m(C)$, and for $m > n > 2$ any two immersions of $P^n(R)$ into $P^m(C)$ which are homotopic are regularly homotopic.

(ii) According to Li [6, p. 257], there exists a null-homotopic immersion of $P^n(R)$ into $P^m(C)$ if and only if there exists an immersion of $P^n(R)$ into R^{2m} .

EXAMPLE. For $n \geq 4$ there exists an immersion of $P^n(R)$ into $P^{n-1}(C)$ if and only if n is not a power of two. If the same condition also holds for $n \geq 7$, then

$$\begin{aligned} I[P^n(R), P^{n-1}(C)] &= I[P^n(R), R^{2n-2}] = \{0\} & n \equiv 0 \pmod{4}, \\ &= Z_2 + Z_2 & n \equiv 1 \pmod{4}, \\ &= Z_2 & n \equiv 2 \pmod{4}, \\ &= Z_4 + Z_8 & n \equiv 3 \pmod{4}, \end{aligned}$$

according to the table in [14].

The proof of Theorem A is given in §2. We shall prove Theorem B along the lines of Li and Habegger [11] (cf. [6]–[7]) in §3.

§2. Proof of Theorem A

Let $x \in H^1(P^n(R); Z_2)$ be the first Stiefel-Whitney class $w_1(\xi)$ of the canonical real line bundle ξ over $P^n(R)$, and $y \in H^2(P^m(C); Z)$ the first Chern class $c_1(\zeta)$ of the canonical complex line bundle ζ over $P^m(C)$. Then the following relations hold (see, e.g., [13]):

$$(2.1) \quad H^*(P^n(R); Z_2) = Z_2[x]/(x^{n+1}), \quad H^*(P^m(C); Z) = Z[y]/(y^{m+1}),$$

$$(2.2) \quad \sum_{i \geq 0} w_i(P^n(R)) = (1+x)^{n+1}, \quad \sum_{j \geq 0} c_j(P^m(C)) = (1-y)^{m+1}.$$

Further we know that

$$(2.3) \quad \rho_2 c_j(P^m(C)) = w_{2j}(P^m(C)) \quad \text{and} \quad \rho_2 y = w_2(\zeta),$$

(see, e.g., [2, Theorem 1.4] or [3, (68)]), and hence

$$(2.4) \quad \sum_{i \geq 0} w_i(P^m(C)) = (1 + \rho_2 y)^{m+1}.$$

It is also easily verified, e.g., by Feder [4], that for a CW-complex X of dimension less than $2m+1$, the correspondence which associates f^*y with a homotopy class of a map $f: X \rightarrow P^m(C)$ leads to a bijection between the homotopy set $[X, P^m(C)]$ and the cohomology group $H^2(X; Z)$. In particular we get

$$(2.5) \quad [P^n(R), P^m(C)] = H^2(P^n(R); Z) = Z_2 = \{\beta_2 x, 0\} \quad \text{if} \quad 2 \leq n \leq 2m.$$

To prove that the natural embedding i is not null-homotopic, it is sufficient to show that $i^*\rho_2y = x^2$. Under the above notations, it is easily seen that the induced bundle $i^*\zeta$ over $P^n(R)$ is isomorphic to the complexification of ζ , i.e.,

$$i^*\zeta = \zeta \otimes C.$$

As a real 2-plane bundle,

$$\zeta \otimes C = \zeta \oplus \zeta$$

(see [13]). Hence, and because $w_2(\zeta \oplus \zeta) = x^2$ and $w_2(\zeta) = \rho_2y$, we have $i^*\rho_2y = x^2$, which establishes Theorem A(i).

Moreover this, together with (2.5) and Remark (i) above, implies that

$$(2.6) \quad [P^n(R), P^m(C)] = \{[i], [c]\} \quad \text{for } 2 \leq n \leq m,$$

where c is a null-homotopic immersion.

To prove (ii), assume that $n \geq 2$ and that there is an immersion $f: P^n(R) \rightarrow P^{n-1}(C)$ such that f is not null-homotopic. Then $f^*\rho_2y = x^2$ by (2.5). Let $w_k(f)$ denote the k -th Stiefel-Whitney class of the normal bundle of this immersion f . Then by the relation $f^*\rho_2y = x^2$ and both by (2.2) and (2.4), we have a relation

$$\sum_{k \geq 0} w_k(f) = (1+x^2)^n(1+x)^{-(n+1)} = \left(\sum_{i \geq 0} \binom{n}{i} x^{2i}\right) \left(\sum_{j \geq 0} \binom{n+j}{n} x^j\right)$$

and in particular

$$(2.7) \quad w_{n-1}(f) = Ax^{n-1}, \quad \text{where } A = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{i} \binom{2n-1-2i}{n}.$$

If $n = 2^r$ ($r \geq 1$), then

$$\binom{n}{i} \equiv \begin{cases} 0 \pmod 2 & \text{for } 0 < i \leq 2^{r-1} - 1 = \lfloor (n-1)/2 \rfloor, \\ 1 \pmod 2 & \text{for } i = 0, \end{cases}$$

and

$$\binom{2n-1}{n} = \binom{2^{r+1}-1}{2^r} \equiv 1 \pmod 2.$$

Hence we have

$$(2.8) \quad A \equiv 1 \pmod 2 \quad \text{if } n = 2^r \ (r \geq 1).$$

Consider, next, the case when $n \neq 2^r$. Then the diadic expansion of n is given by

$$n = \sum_{s \geq i \geq 1} 2^{r_i}, \quad 0 \leq r_1 < r_2 < \dots < r_s, \quad 2 \leq s.$$

Under this expression of n , it is easily seen that $2^{r_{s-1}} \leq [(n-1)/2] < 2^{r_s}$ and that for $0 \leq i \leq [(n-1)/2]$, $\binom{n}{i} \equiv 1 \pmod{2}$ if and only if either $i=0$ or $i = \sum_{j \in J} 2^{r_j}$, $J \subset \{1, 2, \dots, s-1\}$, and hence

$$A \equiv \sum_{J \subset \{1, 2, \dots, s-1\}} \binom{2n - 2 \sum_{j \in J} 2^{r_j} - 1}{n} + \binom{2n-1}{n} \pmod{2}.$$

Here

$$\binom{2n-1}{n} = \binom{\sum_{s \geq j \geq 3} 2^{r_j+1} + 2^{r_2+1} + 2^{r_1+1} - 1}{\sum_{s \geq j \geq 3} 2^{r_j} + 2^{r_2} + 2^{r_1}} \equiv 0 \pmod{2}.$$

If $J = \{1, 2, \dots, s-1\}$, then $i = 2^{r_{s-1}} + \dots + 2^{r_1}$ and

$$\binom{2n-2i-1}{n} = \binom{2^{r_{s-1}}-1}{n} \equiv 1 \pmod{2}.$$

If $J \neq \{1, 2, \dots, s-1\}$, let $k = \min \{j | j \notin J\}$ ($1 \leq k \leq s-1$). Then

$$\binom{2n-2i-1}{n} = \binom{\sum_{s \geq j > k, j \notin J} 2^{r_j+1} + 2^{r_{k+1}} - 1}{\sum_{s \geq j > k} 2^{r_j} + \sum_{k \geq j \geq 1} 2^{r_j}} \equiv 0 \pmod{2}.$$

Therefore

$$(2.9) \quad A \equiv 1 \pmod{2} \quad \text{if } n \neq 2^r.$$

From (2.7)-(2.9), we have

$$w_{n-1}(f) \neq 0,$$

which is a contradiction because the normal bundle of this immersion f is an $(n-2)$ -plane bundle. This implies that immersions of $P^n(R)$ into $P^{n-1}(C)$ are always null-homotopic, which is the case for any immersion of $P^n(R)$ into $P^m(C)$ for $m < n$.

§3. Proof of Theorem B

Let ξ and ζ be m - and n -dimensional real vector bundles ($m < n$) over the spaces X and Y , respectively, and let $\text{Mono}(\xi, \zeta)$ be the space of all vector bundle maps of ξ to ζ which are monomorphisms on each fiber. Moreover for a map $f: X \rightarrow Y$, let $\text{Mono}_f(\xi, \zeta)$ and $\text{Mono}_{[f]}(\xi, \zeta)$ be the subspaces of $\text{Mono}(\xi, \zeta)$ consisting of monomorphisms covering, respectively, f and maps homotopic to f . Then Li and Habegger [11, 3.1] have shown that there is a bijection

$$(3.1) \quad \pi_0(\text{Mono}_f(\xi, \zeta)) = \pi_0(\text{Mono}_{[f]}(\xi, \zeta)) \quad \text{if } \pi_1(Y^X, f) = 0.$$

For a manifold M , let τ_M denote its tangent bundle. The Smale-Hirsch theorem says that if $\dim M < \dim N$, then there is a bijection between the sets $I[M, N]$ and $\pi_0(\text{Mono}(\tau_M, \tau_N))$. In particular we get

$$(3.2) \quad I[M, N]_f = \pi_0(\text{Mono}_{[f]}(\tau_M, \tau_N)).$$

If $N = P^m(C)$ and $\dim M < 2m$, then the Eilenberg classification theorem (see [16, p. 243]) says that

$$\pi_1(P^m(C)^M, f) = H^2(M \times (I, \dot{I}); \pi_2(P^m(C))) = H^1(M; Z).$$

Hence for any $f: P^n(R) \rightarrow P^m(C)$,

$$\pi_1(P^m(C)^{P^n(R)}, f) = 0 \quad \text{if } n < 2m.$$

Therefore for any $f: P^n(R) \rightarrow P^m(C)$, there is a bijection

$$(3.3) \quad I[P^n(R), P^m(C)]_f = \pi_0(\text{Mono}_f(\tau_{P^n(R)}, \tau_{P^m(C)})) \quad \text{if } n < 2m.$$

If $n = m$, we have

$$\begin{aligned} &\pi_0(\text{Mono}_f(\tau_{P^n(R)}, \tau_{P^n(C)})) \\ &= \begin{cases} H^n(P^n(R); Z[f^*w_1(P^n(C)) - w_1(P^n(R))]) & \text{for even } n, \\ H^n(P^n(R); Z_2) & \text{for odd } n, \end{cases} \end{aligned}$$

where $Z[a]$ stands for the integers twisted by $a \in H^1(P^n(R); Z_2)$ (see, e.g., [11, Proposition 4.7.1]). Hence

$$\pi_0(\text{Mono}_f(\tau_{P^n(R)}, \tau_{P^n(C)})) = \begin{cases} Z & \text{if } n \text{ is even,} \\ Z_2 & \text{if } n \text{ is odd.} \end{cases}$$

This, together with (3.3), leads to Theorem B(i).

For a constant map c of $P^n(R)$ to either $P^m(C)$ or R^{2m} , we have

$$\begin{aligned} \pi_0(\text{Mono}_c(\tau_{P^n(R)}, \tau_{P^m(C)})) &= \pi_0(\text{Mono}_1(\tau_{P^n(R)}, P^n(R) \times R^{2m})) \\ &= \pi_0(\text{Mono}_c(\tau_{P^n(R)}, \tau_{R^{2m}})), \end{aligned}$$

because the space $\text{Mono}_f(\xi, \zeta)$ is isomorphic to $\text{Mono}_1(\xi, f^*\zeta)$. Hence using (3.1)–(3.3) and the fact that $\pi_1((R^{2m})^{P^n(R)}, c) = 0$, we have

$$I[P^n(R), P^m(C)]_c = I[P^n(R), R^{2m}]_c = I[P^n(R), R^{2m}].$$

This, together with Theorem A(ii), completes the proof of Theorem B(ii).

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