Probabilities that Random Spherical Caps are in Contact

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Abstract

Consider four random spherical caps of common radius on the unit sphere, and assume that their centers, $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, $\mathbf{p}_4$, are generated independently and uniformly. Let G be a graph made of four vertices v_1, v_2, v_3, v_4, for which vertices v_i and v_j are connected by an edge if and only if spherical caps with centers $\mathbf{p}_i, \mathbf{p}_j$ are in-contact. We study the following two events: E_I that G is composed of a connected triangle of three vertices and an isolated vertex; E_{II} that G is composed of a connected line-segment of four vertices. Since both events occur with zero probability, it is impossible to define the ratio $P(E_I) : P(E_{II})$ by the usual manner. However, by introducing a concept of ε-contactness and later letting ε be arbitrarily small, we can invest a legitimacy to the ratio in an asymptotic sense. On this foundation an exact expression for the ratio will be derived and then it will be evaluated numerically for various radii of speherical caps.

Keywords : spherical cap, zero probability

1 Main result

Random spherical caps produce many interesting problems of both geometrical and probabilistic nature. Investigation on classical packing and covering problems has begun a long time ago (see Fejes Tóth (1972)). Ever since a new kind of packing and covering problems have been continuously proposed and studied (see Maehara (1988), Fejes Tóth (1999), Sugimoto and Tanemura (2001) , Maehara (2004)). In this paper we study an another kind of problem.

Consider random spherical caps on the unit sphere. Let $C(\mathbf{p}, r)$ denote a spherical cap with center $\mathbf{p}$ and radius r (Throughout the paper r is fixed).

Let us denote the spherical distance between two points $\mathbf{p}, \mathbf{q}$ by $\rho(\mathbf{p}, \mathbf{q})$. We say that two spherical caps $C(\mathbf{p}, r)$ and $C(\mathbf{q}, r)$ are in ε-contact if and only if $2r \leq \rho(\mathbf{p}, \mathbf{q}) < 2r + \varepsilon$.

Consider random spherical caps $\{C(\mathbf{p}_1, r), C(\mathbf{p}_2, r), C(\mathbf{p}_3, r), C(\mathbf{p}_4, r)\}$, for which we assume that a set of points $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4\}$ are generated at random, that is, generated independently and uniformly. Let E_{ij} stand for the event that spherical caps $C(\mathbf{p}_i, r)$ and $C(\mathbf{p}_j, r)$ are ε-contact, and E'_{ij} stand for its complement.

In this paper we study probabilities for several events such that some of random spherical caps are in contact. To speak exactly, we first consider events such that some of random spherical caps are in ε-contact, and later find probabilities for these events when ε is infinitesimal.

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To simplify the notation we write $R_0 = 2r$, $R_\varepsilon = 2r + \varepsilon$ and $c = \cos R_0$, $s = \sin R_0$. Also write simply $\rho_{ij} = \rho(\mathbf{p}_i, \mathbf{p}_j)$. Sometimes it is convenient to introduce variables t_{ij} defined by $t_{ij} = \rho_{ij} - R_0$.

In our study two angles $\alpha, \beta(\varphi)$ play much important role, where α denotes one of three interior angles of a regular triangle with side R_0, and $\beta(\varphi)$ denotes one of two interior angles of a rhombus with side R_0 and the other two angles φ. It is easy to see that

$$\cos\alpha = \frac{c}{1+c} \tag{1}$$

and

$$\cos\beta(\varphi) = -\frac{s^2 + (1+c^2)\cos\varphi}{1 + c^2 + s^2\cos\varphi}. \tag{2}$$

Consider a graph G of four vertices v_1, v_2, v_3, v_4, for which we connect vertices v_i and v_j by an edge if and only if spherical caps $C(\mathbf{p}_i, r)$ and $C(\mathbf{p}_j, r)$ are in ε-contact. We study the following two events: E_I that G is composed of a connected triangle of three vertices and an isolated vertex; E_{II} that G is composed of a connected line-segment of four vertices. Define functions

$$\begin{aligned} h_I(r) &= \frac{(2\pi - 3\alpha)(1+3c)s}{4\pi^2 \sin\alpha}, \\ h_{II}(r) &= \frac{3s^3}{2\pi^2}\left[\frac{11}{6}\pi^2 - 4\pi\alpha + 3\alpha^2 - 2\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\frac{1-c}{1+c}\right)^n\right]. \end{aligned}$$

Theorem. *Assume that $r \le \pi/4$. Then*

$$P(E_I) \approx h_I(r)\,\varepsilon^3, \quad P(E_{II}) \approx h_{II}(r)\,\varepsilon^3.$$

By numerical evaluation we see that $h_I(r) > h_{II}(r)$ if $r < 0.0851179\cdots$ and $h_I(r) < h_{II}(r)$ otherwise. Furthermore, since $h_{II}(r)$ tends to zero as r decreases to zero, it is plausible to imagine that in the infinite plane the event E_{II} never occur even if the event E_I can happen by some unknown mechanism.

2 Proof

Lemma 1.

$$P(E_{12}) \approx \frac{s}{2}\,\varepsilon.$$

(Proof) Without loss of generality we may fix $\mathbf{p}_1$. Then we have

$$P(E_{12}) = P\left(\mathbf{p}_2 \in C(\mathbf{p}_1, 2r+\varepsilon) \setminus C(\mathbf{p}_1, 2r)\right) = P(\mathbf{p}_2 \in C(\mathbf{p}_1, 2r+\varepsilon)) - P(\mathbf{p}_2 \in C(\mathbf{p}_1, 2r)).$$

It is well-known that the area of a spherical cap $C(\mathbf{p}, r)$, which we denote by $|C(\mathbf{p}, r)|$, is given by $2\pi(1 - \cos r)$. Therefore we obtain

$$P(E_{12}) = \frac{1}{2}\left(\cos 2r - \cos(2r+\varepsilon)\right) \approx \frac{s}{2}\,\varepsilon.$$

(Q.E.D.)

Lemma 2. *Suppose that* t_{12} *is less than* ε. *Then*

$$P(E_{13} \cap E_{23} | \rho_{12} \text{ is given}) \approx \frac{1}{2\pi \sin\alpha}\,\varepsilon^2.$$

(Proof) Without loss of generality we may fix $\mathbf{p}_1$ and $\mathbf{p}_2$. Draw line segments $\mathbf{p}_1\mathbf{p}_2, \mathbf{p}_2\mathbf{p}_3$ and denote the interior angle of $\angle \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3$ in the triangle $\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3$ by φ. Then the desired conditional probability, which we simply write by P, can be expressed as

$$P = \int_{R_0 \le \rho_{23} < R_\varepsilon, R_0 \le \rho_{13} < R_\varepsilon, 0 \le \varphi < 2\pi} \frac{\sin\rho_{23} d\rho_{23} d\varphi}{4\pi} \approx \frac{s}{2\pi} \int_{0 \le t_{23} < \varepsilon, R_0 \le \rho_{13} < R_\varepsilon, 0 \le \varphi < \pi} dt_{23} d\varphi.$$

Now, in the triangle $\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3$, it holds that

$$\cos\rho_{13} = \cos\rho_{12}\cos\rho_{23} + \sin\rho_{12}\sin\rho_{23}\cos\varphi).$$

Furthermore, since ε is infinitesimal, we have approximations

$$\cos\rho_{12} \approx c - st_{12},\ \sin\rho_{12} \approx s + ct_{12}$$

and similar approximations for $\cos\rho_{23}, \sin\rho_{23}$. Hence it follows

$$\cos\rho_{13} \approx [c^2 - cs(t_{12} + t_{23})] + [s^2 + cs(t_{12} + t_{23})]\cos\varphi.$$

Accordingly we can see

$$R_0 < \rho < R_\varepsilon \Leftrightarrow \cos\alpha\left(1 + \frac{t_{12} + t_{23}}{s}\right) > \cos\varphi > \cos\alpha\left(1 + \frac{t_{12} + t_{23}}{s}\right) - \frac{\varepsilon}{s}.$$

If we put $\varphi = \alpha + \delta$, where δ is infinitesimal, we have $\cos\varphi \approx \cos\alpha - \delta\sin\alpha$, which implies that

$$R_0 < \rho < R_\varepsilon \Leftrightarrow -\frac{\cos\alpha}{s\sin\alpha}(t_{12} + t_{23}) < \delta < -\frac{\cos\alpha}{s\sin\alpha}(t_{12} + t_{23}) + \frac{\varepsilon}{s\sin\alpha}.$$

Therefore

$$P \approx \frac{s}{2\pi}\int_0^\varepsilon dt_{23} \int_{R_0 \le \rho_{13} < R_\varepsilon, 0 \le \varphi < \pi} d\varphi \approx \frac{s}{2\pi}\int_0^\varepsilon dt_{23}\,\frac{\varepsilon}{s\sin\alpha} = \frac{1}{2\pi\sin\alpha}\,\varepsilon^2.$$

(Q.E.D.)

Lemma 3. *Suppose that all* t_{12}, t_{23}, t_{13} *are less than* ε. *Then*

$$P(E_{12} \cap E_{13} \cap E_{23}) \approx \frac{s}{4\pi\sin\alpha}\,\varepsilon^3.$$

(Proof)

$$\begin{aligned} & P(E_{12} \cap E_{13} \cap E_{23}) \\ = & \int_0^\varepsilon P(E_{13} \cap E_{23} | \rho_{12} = R_0 + t)\, P(R_0 + t \le \rho_{12} < R_0 + t + dt) \\ \approx & \int_0^\varepsilon \frac{\varepsilon^2}{2\pi\sin\alpha}\, P(R_0 + t \le \rho_{12} < R_0 + t + dt) \\ = & \frac{\varepsilon^2}{2\pi\sin\alpha}\, P(E_{12}) \approx \frac{s}{4\pi\sin\alpha}\,\varepsilon^3. \end{aligned}$$

Lemma 4. *Suppose that all* t_{12}, t_{23}, t_{13} *are less than* ε. *Then*

$$P\left(E'_{14} \cap E'_{24} \cap E'_{34} | \rho_{12}, \rho_{23}, \rho_{13} \text{ are given}\right) \approx \frac{(2\pi - 3\alpha)(1+3c)}{4\pi}.$$

(Proof)　If we consider a domain

$$D_{123} = C(\mathbf{p}_1, R_\varepsilon) \cup C(\mathbf{p}_2, R_\varepsilon) \cup C(\mathbf{p}_3, R_\varepsilon),$$

then the desired conditional probability, which we write P, is given by

$$P = 1 - P(\mathbf{p}_4 \in D_{123}) = 1 - \frac{|D_{123}|}{4\pi}.$$

Define the point $\mathbf{q}_1$ as the intersection of circles $\partial C(\mathbf{p}_2, R_\varepsilon) \cap \partial C(\mathbf{p}_3, R_\varepsilon)$ that lies in the oppsite side to $\mathbf{p}_1$ with respect to the line segment $\mathbf{p}_2\mathbf{p}_3$. Similarly we define $\mathbf{q}_2, \mathbf{q}_3$ (See the figure in the below). Consider a subset of a cap $C(\mathbf{p}_1, R_\varepsilon)$ which is enclosed by two radii $\mathbf{p}_1\mathbf{q}_2, \mathbf{p}_1\mathbf{q}_3$ and one arc $\mathbf{q}_3\mathbf{q}_2$ and which does not contain the triangle $\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3$, and call it a fan $F(\mathbf{p}_1, R_\varepsilon, \angle \mathbf{q}_2\mathbf{p}_1\mathbf{q}_3)$. Then we see that the region D is composed of four triangles

$$\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3,\ \mathbf{q}_1\mathbf{p}_3\mathbf{p}_2,\ \mathbf{q}_2\mathbf{p}_1\mathbf{p}_3,\ \mathbf{q}_3\mathbf{p}_2\mathbf{p}_1$$

and three fans

$$F(\mathbf{p}_1, R_\varepsilon, \angle \mathbf{q}_2\mathbf{p}_1\mathbf{q}_3), F(\mathbf{p}_2, R_\varepsilon, \angle q_3\mathbf{p}_2\mathbf{q}_3), F(\mathbf{p}_3, R_\varepsilon, \angle \mathbf{q}_1\mathbf{p}_3\mathbf{q}_2).$$

It is obvious that, as ε is infinitesimal,

$$|D_{123}| \approx 4\,|T_0| + 3\,|F_0|,$$

where T_0 denotes a regular triangle with side R_0 and interior angle α, and F_0 denotes a fan $C(\mathbf{p}_1, R_0, 2\pi - 3\alpha)$. Therefore

$$|D_{123}| \approx 4\,(3\alpha - \pi) + 3\,(2\pi - 3\alpha)(1-c),$$

from which immediately follows the conclusion.

(Q.E.D.)

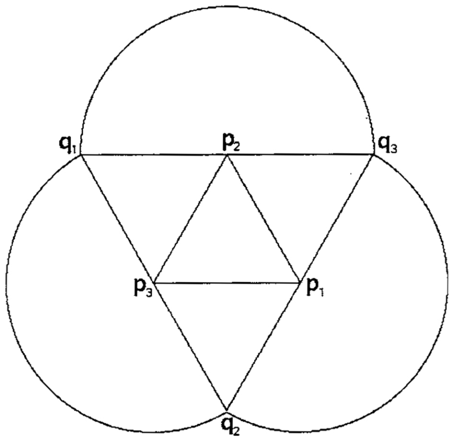

Figure 1

Lemma 5.

$$P(E_{12} \cap E_{13} \cap E_{23} \cap E'_{14} \cap E'_{24} \cap E'_{34}) \approx \frac{(2\pi - 3\alpha)(1+3c)s}{16\pi^2 \sin\alpha} \varepsilon^3.$$

(Proof) The desired probability, which we write simply by P, is given by

$$P = \int_0^{\varepsilon}\int_0^{\varepsilon}\int_0^{\varepsilon} P\left(E'_{14} \cap E'_{24} \cap E'_{34} | \rho_{12} = R_0 + t_{12}, \rho_{23} = R_0 + t_{23}, \rho_{13} = R_0 + t_{13}\right) dW,$$

where dW denotes an infinitesimal probability

$$P(R_0+t_{12} \le \rho_{12} < R_0+t_{12}+dt_{12}, R_0+t_{23} \le \rho_{23} < R_0+t_{23}+dt_{23}, R_0+t_{13} \le \rho_{13} < R_0+t_{13}+dt_{13}).$$

Then Lemma 3 and Lemma 4 imply

$$\begin{aligned} P &\approx \frac{(2\pi - 3\alpha)(1+3c)}{4\pi} \int_0^{\varepsilon}\int_0^{\varepsilon}\int_0^{\varepsilon} dW = \frac{(2\pi - 3\alpha)(1+3c)}{4\pi} P(E_{12} \cap E_{13} \cap E_{23}) \\ &\approx \frac{(2\pi - 3\alpha)(1+3c)}{4\pi} \cdot \frac{s}{4\pi \sin\alpha} \varepsilon^3 \end{aligned}.$$

(Q.E.D.)

In the following lemma we study the conditional probability that $C(\mathbf{p}_4, r)$ is in ε-contact with $C(\mathbf{p}_3, r)$, but neither with $C(\mathbf{p}_1, r)$ nor $C(\mathbf{p}_2, r)$. To state exactly, under the condition

$$H = \{\text{both } \rho_{12} \text{ and } \text{ are given}, \rho_{13} > R_\varepsilon, \angle \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3 = \varphi\},$$

we study the conditional probability

$$P(t_{12}, t_{23}, \varphi) := P(\mathbf{p}_4 \in A(\mathbf{p}_3, R_0, R_\varepsilon) \setminus D_{12} \mid H),$$

where

$$A(\mathbf{p}_3, R_0, R_\varepsilon) = C(\mathbf{p}_3, R_\varepsilon) \setminus C(\mathbf{p}_3, R_0), \quad D_{12} = C(\mathbf{p}_1, R_\varepsilon) \cup C(\mathbf{p}_2, R_\varepsilon).$$

Lemma 6. *Suppose that both* t_{12} *and* t_{23} *are less than* ε. *Furthermore suppose that* $\varphi \ge 0$. *Then there are thresholds* $\varphi_1, \varphi_2, \varphi_3$ *, which are approximately*

$$\varphi_1 \approx 2\alpha, \ \varphi_1 - \varphi_2 \approx O(\varepsilon), \ \varphi_3 \approx \alpha, \tag{3}$$

such that
(1) if $\pi \ge \varphi > \varphi_1$,

$$P(t_{12}, t_{23}, \varphi) \approx \frac{(2\pi - 2\alpha)s\varepsilon}{4\pi}, \tag{4}$$

(2) if $\varphi_2 > \varphi > \varphi_3$,

$$P(t_{12}, t_{23}, \varphi) \approx \frac{(2\pi - \alpha - \beta(\varphi))s\varepsilon}{4\pi}, \tag{5}$$

(3) if $\varphi_3 > \varphi$, *the condition* H *can not be satisfied.*

(Proof)
Step 1 Let $\mathbf{q}_0$ be the intersection of two circles $\partial C(\mathbf{p}_1, R_\varepsilon)$ and $\partial C(\mathbf{p}_2, R_\varepsilon)$. Let $\mathbf{q}_1$ be the

nearer one to $\mathbf{q}_0$ among two intersections of the circle $\partial C(\mathbf{p}_3, R_0)$ with ∂D_{12}, and similarly q_2 the nearer one to $\mathbf{q}_0$ among two intersections of the circle $\partial C(\mathbf{p}_3, R_\varepsilon)$ with ∂D_{12}. The points $\mathbf{q}_1, \mathbf{q}_2$ move on ∂D_{12} as φ decreases from π to 0. When φ, decreasing from π, coincides with some value φ_1, the point $\mathbf{q}_2$ coincides with $\mathbf{q}_0$. Further when φ continues to decrease and becomes some value $\varphi_2(< \varphi_1)$, the point $\mathbf{q}_1$ coincides with $\mathbf{q}_0$. Finally, when φ becomes some value $\varphi_3(< \varphi_2)$, the point $\mathbf{p}_3$ transverses the circle $\partial C(\mathbf{p}_1, R_\varepsilon)$. In step 1 we will determine these thresholds $\varphi_1, \varphi_2, \varphi_3$.

To determine φ_1, draw triangles $\mathbf{p}_1\mathbf{p}_2\mathbf{q}_0$ and $\mathbf{p}_2\mathbf{p}_3\mathbf{q}_0$ (See Figure 2). Since $\rho(\mathbf{p}_1, \mathbf{q}_0) = \rho(\mathbf{p}_2, \mathbf{q}_0) = \rho(\mathbf{p}_3, \mathbf{q}_0) = R_\varepsilon$ and $t_{12} < \varepsilon, t_{23} < \varepsilon$, we see that both the triangles are nearly regular triangles with side R_0. Therefore we can deduce

$$\varphi_1 = 2\alpha + O(\varepsilon). \tag{6}$$

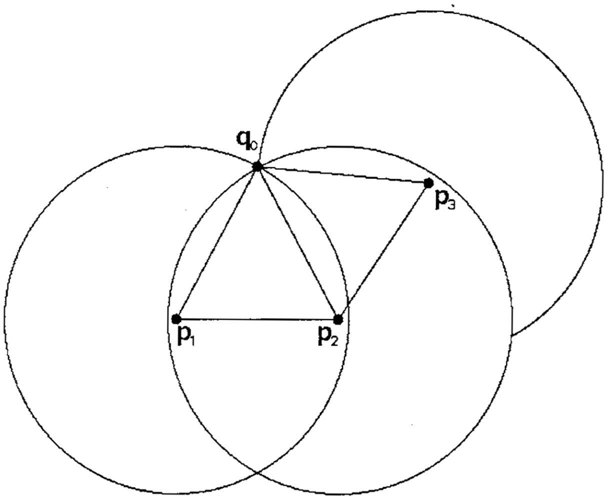

Figure 2

To determine φ_2, we consider the triangles $\mathbf{p}_1\mathbf{p}_2\mathbf{q}_0$ and $\mathbf{p}_2\mathbf{p}_3\mathbf{q}_0$ again (See Figure 2 again). Although in this case $\rho(\mathbf{p}_3, \mathbf{q}_0) = R_0$, both the triangles are again nearly regular triangles Accordingly we get $\varphi_2 = 2\alpha + O(\varepsilon)$ and therefore

$$\varphi_1 - \varphi_2 = O(\varepsilon). \tag{7}$$

To determine φ_3, we consider a triangle $\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3$. Since its sides are $\rho_{12}, \rho_{23}, R_\varepsilon$, it is nearly a regular triangle. Therefore we get

$$\varphi_3 = \alpha + O(\varepsilon). \tag{8}$$

Step 2 In this step we will obtain an expression for an area $|A(\mathbf{p}_3, R_0, R_\varepsilon)|$.

First consider the case that $\varphi > \varphi_1$. Then the circle $\partial C(\mathbf{p}_3, R_0)$ intersects with ∂D_{12} at two points on the circle $\partial C(\mathbf{p}_2, R_\varepsilon)$. Let $\mathbf{q}$ be one of these intersections that lies nearer to $\mathbf{q}_0$. (See Figure 3, where $\rho(\mathbf{p}_3, \mathbf{q}) = R_0$). Since the triangle $\mathbf{p}_2\mathbf{p}_3\mathbf{q}$ is nearly a regular triangle, we see $\psi := \angle \mathbf{p}_2\mathbf{p}_3\mathbf{q} \approx \alpha$. Note that an angle at $\mathbf{p}_3$ that corresponds to the region $A(\mathbf{p}_3, R_0, R_\varepsilon)$ is equal to $2\pi - 2\psi$. Therefore we obtain

$$|A(\mathbf{p}_3, R_0, R_\varepsilon)| \approx (2\pi - 2\alpha)s\varepsilon,$$

from which (4) immediately follows.

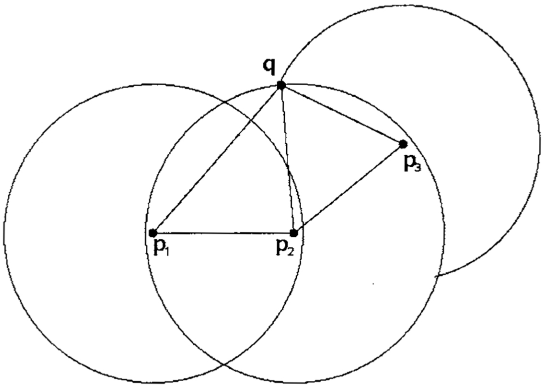

Figure 3

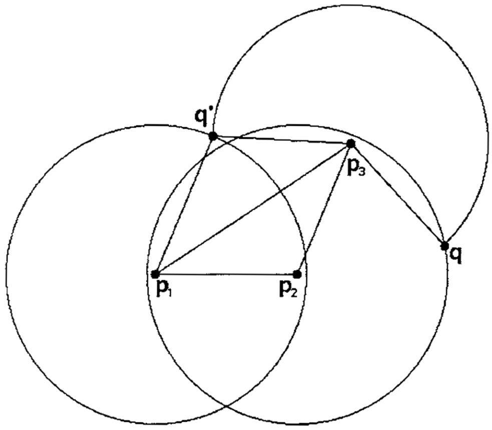

Figure 4

Next consider the case that $\varphi_2 > \varphi > \varphi_3$. Then the circle $\partial C(\mathbf{p}_3, R_0)$ intersects with ∂D_{12} at two points, one of which lies on the circle $\partial C(\mathbf{p}_2, R_\varepsilon)$ and the other lies on the circle $\partial C(\mathbf{p}_1, R_\varepsilon)$. Let $\mathbf{q}$ be the intersection lying on the circle $\partial C(\mathbf{p}_2, R_\varepsilon)$, and $\mathbf{q}'$ the intersection lying on the circle $\partial C(\mathbf{p}_1, R_\varepsilon)$ (See Figure 4).

Then an angle $\psi := \angle \mathbf{p}_2\mathbf{p}_3\mathbf{q}$ is again nearly equal to α. On the other hand, a quadrangle $\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\mathbf{q}'$ is nearly a rhombus with side R_0 and $\angle \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3 = \varphi$. Accordingly we see that $\psi' := \angle \mathbf{p}_2\mathbf{p}_3\mathbf{q}' \approx \beta(\varphi)$. Note that an angle at $\mathbf{p}_3$ that corresponds to the region $A(\mathbf{p}_3, R_0, R_\varepsilon)$ is equal to $2\pi - (\psi' + \psi)$. Therefore we obtain

$$|A(\mathbf{p}_3, R_0, R_\varepsilon)| \approx (2\pi - \alpha - \beta(\varphi))s\varepsilon,$$

from which (5) immediately follows.

(Q.E.D.)

Lemma 7.

$$\begin{aligned} &P(E_{12} \cap E_{23} \cap E_{34} \cap E'_{13} \cap E'_{14} \cap E'_{24}) \\ \approx\ & \frac{(s\varepsilon)^3}{16\pi^2} \left[\frac{11}{6}\pi^2 - 4\pi\alpha + 3\alpha^2 - 2\sum_{n=1}^{\infty} \frac{1}{n^2} \left(\frac{1-c}{1+c} \right)^n \right] \end{aligned}$$

(Proof) Without loss of generality we may fix $\mathbf{p}_1$. The desired probability, which we denote by P, can be computed by

$$P = 2\int_0^\varepsilon \int_0^\varepsilon \int_{\varphi_3}^\pi P(t_{12}, t_{23}, \varphi) \cdot \frac{2\pi \sin(2r + t_{12})dt_{12}}{4\pi} \cdot \frac{\sin(2r + t_{23})dt_{23}d\varphi}{4\pi}$$

$$\begin{aligned} &= \frac{1}{4\pi}\int_0^{\varepsilon}\sin(2r+t_{12})dt_{12}\int_0^{\varepsilon}\sin(2r+t_{23})dt_{23}\int_{\varphi_3}^{\pi}P(t_{12},t_{23},\varphi)\,d\varphi. \\ &\approx \frac{(s\varepsilon)^2}{4\pi}\int_{\varphi_3}^{\pi}P(0,0,\varphi)\,d\varphi. \end{aligned}$$

The last integral is the sum of

$$I_1=\int_{\varphi_1}^{\pi}P(0,0,\varphi)\,d\varphi,\ I_2=\int_{\varphi_2}^{\varphi_1}P(0,0,\varphi)\,d\varphi,\ I_3=\int_{\varphi_3}^{\varphi_2}P(0,0,\varphi)\,d\varphi.$$

By using (3) we see easily $I_2=O\left(\varepsilon^2\right)$. On the other hand, (4) implies

$$I_1\approx\frac{\pi-\alpha}{2\pi}s\varepsilon\cdot(\pi-2\alpha)$$

and (5) implies

$$I_3\approx\int_{\varphi_3}^{\varphi_2}\frac{2\pi-\alpha-\beta(\varphi)}{4\pi}\,s\varepsilon\,d\varphi\approx\frac{s\varepsilon}{4\pi}\left[(2\pi-\alpha)\cdot\alpha-\int_{\alpha}^{2\alpha}\beta(\varphi)\,d\varphi\right].$$

Therefore

$$P\approx\frac{(s\varepsilon)^3}{16\pi^2}\left[(2\pi^2-4\pi\alpha+3\alpha^2)-\int_{\alpha}^{2\alpha}\beta(\varphi)\,d\varphi\right].$$

Thus it remains to evaluate the last integral.

Now we regard the last integral as a function of c:

$$J(c):=\int_{\alpha}^{2\alpha}\beta(\varphi)\,d\varphi.$$

To differentiate it, then we get

$$\frac{\partial J}{\partial c}=2\,\frac{\partial\alpha}{\partial c}\cdot\beta(2\alpha)-\frac{\partial\alpha}{\partial c}\cdot\beta(\alpha)+\int_{\alpha}^{2\alpha}\frac{\partial\beta}{\partial c}\,d\varphi.$$

Since

$$\beta(\alpha)=2\alpha,\ \beta(2\alpha)=\alpha$$

and

$$\frac{\partial\beta}{\partial c}=-\frac{2\sin\varphi}{1+c^2+s^2\cos\varphi},$$

we have

$$\frac{\partial J}{\partial c}=-2\int_{\alpha}^{2\alpha}\frac{\sin\varphi}{1+c^2+s^2\cos\varphi}\,d\varphi.$$

The last integral can be evaluated to

$$\frac{\partial J}{\partial c}=\frac{2}{s^2}\log\frac{1+c^2+s^2\cos2\alpha}{1+c^2+s^2\cos\alpha}.$$

Then, since

$$\frac{1+c^2+s^2\cos2\alpha}{1+c^2+s^2\cos\alpha}=\left(\frac{2c}{1+c}\right)^2,$$

we get

$$\frac{\partial J}{\partial c} = \frac{4}{1-c^2} \log\left(\frac{2c}{1+c}\right).$$

Note that, when $c = 1$, we have $\alpha = \pi/3, \beta(\varphi) = \pi - \varphi$. Hence

$$J(1) = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} (\pi - \varphi)\, d\varphi = \frac{\pi^2}{6}.$$

On the other hand, by change of variable $c = (1-x)/(1+x)$, we have

$$\begin{aligned} J(c) - J(1) &= \int_1^c \frac{4}{1-c^2} \log\left(\frac{2c}{1+c}\right) dc = -2 \int_0^x \log(1-x)\, \frac{dx}{x} \\ &= 2\sum_{n=1}^{\infty} \frac{1}{n} \int_0^x x^{n-1}\, dx = 2\sum_{n=1}^{\infty} \frac{x^2}{n^2}. \end{aligned}$$

Therefore we obtain

$$J(c) = \frac{\pi^2}{6} + 2\sum_{n=1}^{\infty} \frac{1}{n^2} \left(\frac{1-c}{1+c}\right)^n.$$

(Q.E.D.)

(Proof of Theorem) When the event E_I occurs, there are four possibilities, one of which is that three vertices v_1, v_2, v_3 are connected each other, but v_4 is isolated. Therefore the probability $P(E_I)$ is equal to four times the probability

$$P(E_{12} \cap E_{13} \cap E_{23} \cap E'_{14} \cap E'_{24} \cap E'_{34}).$$

When the event E_{II} occurs, there are 4! possibilities, one of which is that there are only three edges v_1v_2, v_2v_3, v_3v_4. Therefore the probability $P(E_{II})$ is equal to 4! times the probability

$$P(E_{12} \cap E_{23} \cap E_{34} \cap E'_{13} \cap E'_{14} \cap E'_{24}).$$

(Q.E.D.)

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