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Numerical Calculation of Tide in Kagoshima Bay Part 2. Two-dimensional Explicit Weighted Residual Method

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Abstract

The tidal residual flow in Kagoshima bay is evaluated by employing the two-dimensional explicit weighed residual method for space, which conserves the water mass, and the two-step Lax-Wendroff method for time differentiation. Water surface elevation is given at the open boundary and the non-slip boundary condition is adopted at the coast. In order to obtain stable results, calculation is performed over five period.

The estimated tidal residual flow shows that there exists a large anti-clockwise vortex at the center of the bay and that the water mass inflows from the east side and outflows from the west side of the mouth of the bay. In spite of the roughness of the division of the bay, the above two features of the results are similar to the observational results so far performed.

1. Introduction

In the previous paper[1], the tide in Kagoshima bay is calculated by making use of the primitive lumped mass marix method for two-dimensional horizontal space. The primitive lumped mass matrix technique is, however, known[2] to be inadequate for the estimation of tidal residual flow because of the appearance of the energy loss in the method, although the resulted tidal flow itself might be almost available[3]. In order to improve this defect of the primitive lumped mass matrix method, Kawahara et al[4] proposed a selective lumping mass matrix method and calculated the tidal residual flow of Osaka bay. In the selective lumping method, however, the time step has to be chosen to be smaller to avoid divergence in the calculation and then the calculation time needed becomes longer. Anyhow, the selective lumping technique is more artificial than the primitive lumped mass matrix method.

For the purpose to decrease the energy loss in the calculation, we have proposed[5] an another explicit method named explicit weighted residual method or EWM for short. EWM is applied in this paper for the calculation of the tidal residual flow in Kagoshima bay. In the resulted tidal residual flow, the loss of water mass is 11.4 % at the open

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boundary and 3.8 % at the center of the bay, which may be admissible if the roughness of the division of the bay is taken into account. There appears a large anti-clockwise vortex at the center of the bay and the water mass inflows from the east side and outflows from the west side of the mouth of the bay. These two features agree qualitatively to observations so far performed.

Some explanations of EWM are given in section 2 for the completeness of this paper. In section 3 are presented the finite element formulations of Euler's equations of motion and equation of continuity, employing EWM for space and two-step Lax-Wendroff scheme for time differentiation. The resulted tidal velocity, tidal mass transport, tidal residual velocity, tidal residual mass transport etc. are shown in section 4. Section 5 is devoted to some discussions about the further improvement.

2. Explicit weighted residual method

Although EWM can be formulated also for the three dimensional and/or complex element, only the two-dimensional simplex element is considered in this paper. In the two-dimensional simplex element, any physical quantity $q = q(x_1, x_2, t)$ is linearly approximated as[6]

$$q(x_1, x_2, t) = L_\alpha(x_1, x_2)q_\alpha(t), \qquad (\alpha = 1\sim3) \tag{1}$$

where q_α denote the values of q at the α-th vertex of the triangle element and L_α are the area coordinates. In Eq. (1) the Einstein's summation convention is used for simplicity, i. e., the repeated Greek indices denote the summation over $1\sim3$.

The weighting function $q^*(x_1, x_2)$ is taken to be the general linear function of coordinates (x_1, x_2) and then written as

$$q^*(x_1, x_2) = N_\alpha(x_1, x_2)q_\alpha^*, \qquad (\alpha = 1\sim3) \tag{2}$$

where q_α^* are arbitrary constants fixed at the α-th vertex of the triangle element and $N_\alpha(x_1, x_2)$ are also arbitrary linear functions of coordinates (x_1, x_2). The mass matrix M is then defined by

$$M_{\alpha\beta} = \int_\Omega N_\alpha L_\beta dA, \tag{3}$$

where Ω is the domain of the simplex element and dA denotes the area integral. For the finite element method to be explicit for space, the mass matrix has to be a diagonal matrix. In case of the Galerkin's method, however, the arbitrary linear functions N_α are identified as the area coordinates L_α and then the mass matrix is found to be

$$M = \frac{A}{12}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \tag{4}$$

which is not a diagonal one. In Eq. (4), A denotes the area of the triangle element. In order to make the method explicit for space, the non-diagonal mass matrix of Eq. (4) is artificially changed to the following diagonal matrix $\bar{M}$ in the lumped mass matrix method

$$\bar{M} = \frac{A}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{5}$$

General linear functions of coordinates N_α can be written by using the area coordinates L_α as

$$N_\alpha = a(\alpha)L_\alpha + b(\alpha)L_\beta + c(\alpha)L_\gamma, \qquad (\alpha,\ \beta,\ \gamma \text{ cyclic}) \tag{6}$$

where $a(\alpha)$, $b(\alpha)$, $c(\alpha)$ are arbitrary constants. The requirement of the diagonality of the mass matrix, i. e. $M_{\alpha\beta} = 0$ for $\alpha \neq \beta$, imposes the following two constraints among the constants $a(\alpha)$, $b(\alpha)$, and $c(\alpha)$,

$$\begin{aligned} a(\alpha)+2b(\alpha)+c(\alpha) &= 0, \\ a(\alpha)+b(\alpha)+2c(\alpha) &= 0. \end{aligned} \tag{7}$$

Then $b(\alpha)$ and $c(\alpha)$ can be solved to be

$$b(\alpha) = c(\alpha) = -\frac{1}{3}a(\alpha). \tag{8}$$

The weighting function, which gives the diagonal mass matrix, is uniquely determined from Eqs. (2), (6) and (8) as

$$q^*(x_1,\ x_2) = (3L_1 - L_2 - L_3)\tilde{q}_1^* + (-L_1 + 3L_2 - L_3)\tilde{q}_2^* + (-L_1 - L_2 + 3L_3)\tilde{q}_3^*, \tag{9}$$

where $\tilde{q}_\alpha^*$ are defined by $a(\alpha)q_\alpha^*/3$. The diagonal mass matrix corresponding to the weighting function of Eq. (9) is written as

$$M = \frac{A}{9}\begin{pmatrix} a(1) & 0 & 0 \\ 0 & a(2) & 0 \\ 0 & 0 & a(3) \end{pmatrix}. \tag{10}$$

Formulating the governing equations by the usual weighted residual method with the weighting function of Eq. (9) and considering $\tilde{q}_{\alpha}^{*}(\alpha = 1 \sim N$, N is the total number of the nodal points in the domain) as arbitrary constants, the method becomes explicit for space. For the purpose to make the method also explicit for time, the two-step Lax-Wendroff time difference method is adopted in this paper. The details will be given in the next section. In EWM, the equations to be overlapped are chosen to some combinations of those in the Galerkin's method so that they could be solved explicitly for space.

3. Finite element formulation by EWM

As the conservative form of equations are known[5] to be more adequate for the estimation of tidal residual flow than those of non-conservative form, the following conservative forms of two-dimensional Euler's equations of motion and equation of continuity will be formulated by the weighted residual method,

$$\frac{\partial u_1}{\partial t}+\frac{\partial(u_1 v_j)}{\partial x_j}+g(h+\eta)\frac{\partial \eta}{\partial x_1}-fu_2-\nu\Delta u_1 = 0, \tag{11}$$

$$\frac{\partial u_2}{\partial t}+\frac{\partial(u_2 v_j)}{\partial x_j}+g(h+\eta)\frac{\partial \eta}{\partial x_2}+fu_1-\nu\Delta u_2 = 0, \tag{12}$$

$$\frac{\partial \eta}{\partial t}+\frac{\partial u_j}{\partial x_j} = 0, \tag{13}$$

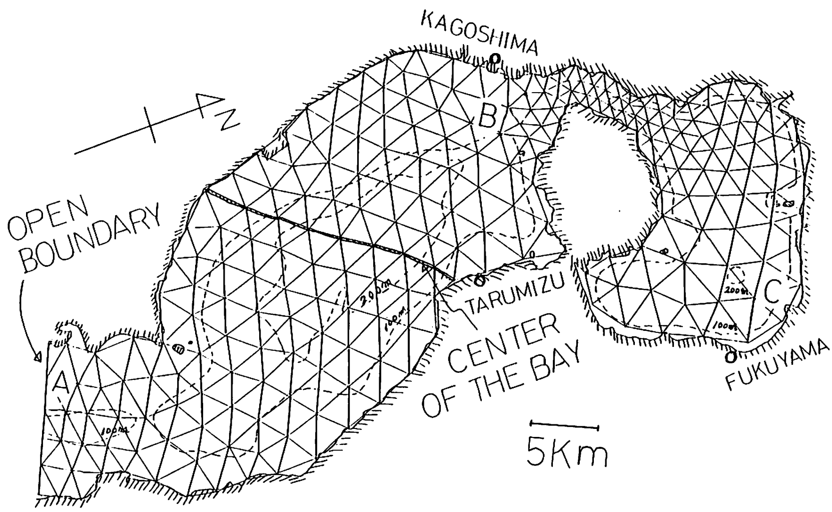


Fig. 1 The division of Kagoshima bay into two dimensional simplex elements.

where v_i ($i = 1, 2$) denote the horizontal velocities, η the water surface elevation, h the depth of the bay, $u_i = (h+\eta)v_i$, g the gravitational acceleration, f the Coriolis parameter and ν is the kinetic viscousity. In Eqs.(11)~(13), the repeated Latin indices denote the summation over 1~2. In order to solve Eqs (11)~(13) in the domain of Kagoshima bay, Kagoshima bay is preliminary divided into the two-dimensional simplex elements as shown in Fig. 1. Considering one of the elements in Fig. 1, the following variational functionals corresponding to Eqs. (11)~(13) are defined

$$\begin{aligned}\delta x^e_{u_1} &\equiv \int_\Omega u_1^*\left[\frac{\partial u_1}{\partial t}+\frac{\partial(u_1 v_j)}{\partial x_j}+g(h+\eta)\frac{\partial \eta}{\partial x_1}-fu_2-\nu\Delta u_1\right]dA \\ &= \int_\Omega\left\{u_1^*\left[\frac{\partial u_1}{\partial t}+\frac{\partial(u_1 v_j)}{\partial x_j}+g(h+\eta)\frac{\partial \eta}{\partial x_1}-fu_2\right]\right. \\ &\quad \left.+\nu\frac{\partial u_1^*}{\partial x_j}\frac{\partial u_1}{\partial x_j}-\nu\frac{\partial}{\partial x_j}\left(u_1^*\frac{\partial u_1}{\partial x_j}\right)\right\}dA, \end{aligned} \tag{14}$$

$$\begin{aligned}\delta x^e_{u_2} &\equiv \int_\Omega\left\{u_2^*\left[\frac{\partial u_2}{\partial t}+\frac{\partial(u_2 v_j)}{\partial x_j}+g(h+\eta)\frac{\partial \eta}{\partial x_2}+fu_1\right]\right. \\ &\quad \left.+\nu\frac{\partial u_2^*}{\partial x_j}\frac{\partial u_2}{\partial x_j}-\nu\frac{\partial}{\partial x_j}\left(u_2^*\frac{\partial u_2}{\partial x_j}\right)\right\}dA, \end{aligned} \tag{15}$$

$$\delta x^e_\eta \equiv \int_\Omega \eta^*\left[\frac{\partial \eta}{\partial t}+\frac{\partial u_j}{\partial x_j}\right]dA. \tag{16}$$

The integrals of the last terms of Eqs. (14) and (15) can be performed with the help of the Gauss' theorem. When all elements are summed up, these terms are cancelled by the neighbouring elements and in the case of the boundary elements, where the neighbouring elements are absent, $u_1{}^*$ and u_2^* are taken to be zero at the boundary coast owing to the non-slip boundary condition adopted in this paper. Thus the last terms of Eqs. (14) and (15) can be cast away in the following.

In our case of EWM, the weighting functions u_1^*, u_2^* and η^* take the form of Eq. (9). Then approximating the physical quantities u_1, u_2 and η as the linear functions of coordinates (see Eq. (1)), variational functionals of one element can be written as

$$\begin{aligned}\delta x^e_{u_1} &= \int_\Omega \tilde{u}_{1\alpha}^*\{\tilde{N}_\alpha[L_\beta \dot{u}_{1\beta}+(L_\beta L_{\gamma,x_j}+L_{\beta,x_j}L_\gamma)u_{1\beta}v_{j\gamma} \\ &\quad +gL_\beta L_{\gamma,x_1}(h_\beta+\eta_\beta)\eta_\gamma-fL_\beta u_{2\beta}]+\nu\tilde{N}_{\alpha,x_j}L_{\beta,x_j}u_{1\beta}\}dA, \end{aligned} \tag{17}$$

$$\begin{aligned}\delta x^e_{u_2} &= \int_\Omega \tilde{u}_{2\alpha}^*\{\tilde{N}_\alpha[L_\beta \dot{u}_{2\beta}+(L_\beta L_{\gamma,x_j}+L_{\beta,x_j}L_\gamma)u_{2\beta}v_{j\gamma} \\ &\quad +gL_\beta L_{\gamma,x_2}(h_\beta+\eta_\beta)\eta_\gamma+fL_\beta u_{1\beta}]+\nu\tilde{N}_{\alpha,x_j}L_{\beta,x_j}u_{2\beta}\}dA, \end{aligned} \tag{18}$$

$$\delta x^e_\eta = \int_\Omega \tilde{\eta}_\alpha^*\tilde{N}_\alpha[L_\beta\dot{\eta}_\beta+L_{\beta,x_j}u_{j\beta}]dA, \tag{19}$$

where

$$\dot{u}_{j\alpha} \equiv \frac{\partial u_{j\alpha}}{\partial t}, \qquad \dot{\eta}_\alpha \equiv \frac{\partial \eta_\alpha}{\partial t},$$

$$L_{\alpha,x_j} \equiv \frac{\partial L_\alpha}{\partial x_j}, \qquad \tilde{N}_{\alpha,x_j} \equiv \frac{\partial \tilde{N}_\alpha}{\partial x_j}, \tag{20}$$

$$\tilde{N}_\alpha \equiv 3L_\alpha - L_\beta - L_\gamma \qquad (\alpha,\ \beta,\ \gamma \text{ cyclic})$$
$$= (V^{-1})_{\alpha\beta} L_\beta, \tag{21}$$

$$(V^{-1}) \equiv \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}. \tag{22}$$

In Eqs. (17) and (18), $v_{j\alpha}$ are taken to be $u_{j\alpha}/(h_\alpha+\eta_\alpha)$.

The method of weighted residuals requires $\delta\chi_{u_1}$, $\delta\chi_{u_2}$ and $\delta\chi_\eta$ defined by

$$\delta\chi_{u_1} \equiv \sum_{\text{elements}} \delta\chi^e_{u_1}, \qquad \delta\chi_{u_2} \equiv \sum_{\text{elements}} \delta\chi^e_{u_2}, \qquad \delta\chi_\eta \equiv \sum_{\text{elements}} \delta\chi^e_\eta, \tag{23}$$

are vanished for any constants $\tilde{u}^*_{1\alpha}$, $\tilde{u}^*_{2\alpha}$ and $\tilde{\eta}^*_\alpha$ for $\alpha = 1\sim N$, where N is the total number of the nodal points in the domain. Instead, we could (i) first require $\delta\chi^e_{u_1}$, $\delta\chi^e_{u_2}$ and $\delta\chi^e_\eta$ are vanished for any constants $\tilde{u}^*_{1\alpha}$, $\tilde{u}^*_{2\alpha}$ and $\tilde{\eta}^*_\alpha$ for $\alpha = 1\sim 3$ of one element obtaining nine equations and then (ii) overlap the nine equations over all elements. The number of the resulted equations is $3\times N$. Solving the $3\times N$ equations, the $3\times N$ unknowns of $u_{1\alpha}$, $u_{2\alpha}$ and η_α for $\alpha = 1 \sim N$ can be resolved.

The nine equations obtained by the requirements of $\delta\chi^e_{u_1} = \delta\chi^e_{u_2} = \delta\chi^e_\eta = 0$ for any constants $\tilde{u}^*_{1\alpha}$, $\tilde{u}^*_{2\alpha}$ and $\tilde{\eta}^*_\alpha(\alpha = 1\sim 3)$ are

$$\tilde{M}_{\alpha\beta}\dot{u}_{1\beta} + \tilde{K}^j_{\alpha\beta\gamma}(u_{1\beta}v_{j\gamma} + v_{j\beta}u_{1\gamma}) + g\tilde{K}^1_{\alpha\beta\gamma}(h_\beta+\eta_\beta)\eta_\gamma$$
$$-f\tilde{M}_{\alpha\beta}u_{2\beta} + \nu\tilde{R}_{\alpha\beta}u_{1\beta} = 0, \tag{24}$$

$$\tilde{M}_{\alpha\beta}\dot{u}_{2\beta} + \tilde{K}^j_{\alpha\beta\gamma}(u_{2\beta}v_{j\gamma} + v_{j\beta}u_{2\gamma}) + g\tilde{K}^2_{\alpha\beta\gamma}(h_\beta+\eta_\beta)\eta_\gamma$$
$$+f\tilde{M}_{\alpha\beta}u_{1\beta} + \nu\tilde{R}_{\alpha\beta}u_{2\beta} = 0, \tag{25}$$

$$\tilde{M}_{\alpha\beta}\dot{\eta}_\beta + \tilde{G}_{\alpha\beta}{}^j u_{j\beta} = 0, \tag{26}$$

where

$$\tilde{M}_{\alpha\beta} \equiv \int_\Omega \tilde{N}_\alpha L_\beta dA = (V^{-1})_{\alpha\rho}\int_\Omega L_\rho L_\beta dA \equiv (V^{-1})_{\alpha\rho}M_{\rho\beta} = \frac{A}{3}\delta_{\alpha\beta},$$

$$\tilde{K}^j_{\alpha\beta\gamma} \equiv \int_\Omega \tilde{N}_\alpha L_\beta L_{\gamma,x_j} dA = (V^{-1})_{\alpha\rho}\int_\Omega L_\rho L_\beta L_{\gamma,x_j} dA \equiv (V^{-1})_{\alpha\rho}K^j_{\rho\beta\gamma},$$

$$\tilde{R}_{\alpha\beta} \equiv \int_\Omega \tilde{N}_{\alpha,x_j} L_{\beta,x_j} dA = (V^{-1})_{\alpha\rho}\int_\Omega L_{\rho,x_j} L_{\beta,x_j} dA \equiv (V^{-1})_{\alpha\rho}R_{\rho\beta},$$

$$\tilde{G}^j_{\alpha\beta} \equiv \int_\Omega \tilde{N}_\alpha L_{\beta,x_j} dA = (V^{-1})_{\alpha\rho}\int_\Omega L_\rho L_{\beta,x_j} dA \equiv (V^{-1})_{\alpha\rho}G^j_{\rho\beta}. \tag{27}$$

The integrals in Eq. (27) can be performed by using the integral formula[7)]

$$\int_\Omega L_1^a L_2^b L_3^c dA = \frac{a!b!c!}{(a+b+c+2)!}2A. \tag{28}$$

Explicit integrations give the form of Eq. (4) for M and

$$K^1_{\alpha\beta\gamma} = \begin{cases} \frac{1}{24}b_\gamma & \text{for } \alpha \neq \beta \\ \frac{1}{12}b_\gamma & \text{for } \alpha = \beta \end{cases} \qquad K^2_{\alpha\beta\gamma} = \begin{cases} \frac{1}{24}c_\gamma & \text{for } \alpha \neq \beta \\ \frac{1}{12}c_\gamma & \text{for } \alpha = \beta \end{cases},$$

$$R_{\alpha\beta} = \frac{1}{4A}(b_\alpha b_\beta + c_\alpha c_\beta),$$

$$G^1_{\alpha\beta} = \frac{1}{6}b_\beta, \qquad G^2_{\alpha\beta} = \frac{1}{6}c_\beta, \tag{29}$$

where

$$b_\alpha = y_\beta - y_\gamma, \qquad c_\alpha = x_\gamma - x_\beta. (\alpha,\ \beta,\ \gamma \text{ cyclic}) \tag{30}$$

In Eq. (30), (x_α, y_α) is the coordinate of the α-th vertex of the triangle element. Eqs. (24) ~ (26) can be rewritten as

$$\frac{A}{3}\dot{u}_{1\alpha} + (V^{-1})_{\alpha\rho}[K^j_{\rho\beta\gamma}(u_{1\beta}v_{j\gamma} + v_{j\beta}u_{1\gamma}) + gK^1_{\rho\beta\gamma}(h_\beta + \eta_\beta)\eta_\gamma$$
$$-fM_{\rho\beta}u_{2\beta} + \nu R_{\rho\beta}u_{1\beta}] = 0, \tag{31}$$

$$\frac{A}{3}\dot{u}_{2\alpha} + (V^{-1})_{\alpha\rho}[K^j_{\rho\beta\gamma}(u_{2\beta}v_{j\gamma} + v_{j\beta}u_{2\gamma}) + gK^2_{\rho\beta\gamma}(h_\beta + \eta_\beta)\eta_\gamma$$
$$+fM_{\rho\beta}u_{1\beta} + \nu R_{\rho\beta}u_{2\beta}] = 0, \tag{32}$$

$$\frac{A}{3}\dot{\eta}_\alpha + (V^{-1})_{\alpha\rho}G^j_{\rho\beta}u_{j\beta} = 0, \tag{33}$$

One of the commonly used explicit scheme for time differentiation is the two-step Lax-Wendroff method. The method implies

$$\dot{q}^n = \frac{q^{n+\frac{1}{2}} - q^n}{\frac{1}{2}\Delta t}, \qquad \dot{q}^{n+\frac{1}{2}} = \frac{q^{n+1} - q^n}{\Delta t}, \tag{34}$$

or

$$q^{n+\frac{1}{2}} = q^n + \frac{1}{2}\Delta t \dot{q}^n,$$
$$q^{n+1} = q^n + \Delta t \dot{q}^{n+\frac{1}{2}} \left(= q^n + \Delta t \dot{q}^n + \frac{1}{2}(\Delta t)^2 \ddot{q}^n \right), \tag{35}$$

where the upper suffices n, $n+\frac{1}{2}$ and $n+1$ denote the time steps. Adopting the two-step Lax-Wendroff method, it finally follows from Eqs. (31)~(33)

$$\frac{A}{3}u_{1\alpha}^{n+\frac{1}{2}} = \frac{A}{3}u_{1\alpha}^{n}+\frac{A}{3}\frac{\Delta t}{2}\dot{u}_{1\alpha}^{n} = \frac{A}{3}u_{1\alpha}^{n} -\frac{\Delta t}{2}(V^{-1})_{\alpha\rho}F_{u_1\rho}^{n}, \tag{36}$$

$$\frac{A}{3}u_{2\alpha}^{n+\frac{1}{2}} = \frac{A}{3}u_{2\alpha}^{n}-\frac{\Delta t}{2}(V^{-1})_{\alpha\rho}F_{u_2\rho}^{n}, \tag{37}$$

$$\frac{A}{3}\eta_{\alpha}^{n+\frac{1}{2}} = \frac{A}{3}\eta_{\alpha}^{n}-\frac{\Delta t}{2}(V^{-1})_{\alpha\rho}F_{\eta\rho}^{n}, \tag{38}$$

$$\frac{A}{3}u_{1\alpha}^{n+1} = \frac{A}{3}u_{1\alpha}^{n}+\Delta t\frac{A}{3}\dot{u}_{1\alpha}^{n+\frac{1}{2}} = \frac{A}{3}u_{1\alpha}^{n}-\Delta t(V^{-1})_{\alpha\rho}F_{u_1\rho}^{n+\frac{1}{2}}, \tag{39}$$

$$\frac{A}{3}u_{2\alpha}^{n+1} = \frac{A}{3}u_{2\alpha}^{n}-\Delta t(V^{-1})_{\alpha\rho}F_{u_2\rho}^{n+\frac{1}{2}}, \tag{40}$$

$$\frac{A}{3}\eta_{\alpha}^{n+1} = \frac{A}{3}\eta_{\alpha}^{n}-\Delta t(V^{-1})_{\alpha\rho}F_{\eta\rho}^{n+\frac{1}{2}}, \tag{41}$$

with

$$F_{u_1\rho} = K_{\rho\beta\gamma}^{j}(u_{1\beta}v_{j\gamma}+v_{j\beta}u_{1\gamma})+gK_{\rho\beta\gamma}^{1}(h_\beta+\eta_\beta)\eta_\gamma - fM_{\rho\beta}u_{2\beta}+\nu R_{\rho\beta}u_{1\beta}, \tag{42}$$

$$F_{u_2\rho} = K_{\rho\beta\gamma}^{j}(u_{2\beta}v_{j\gamma}+v_{j\beta}u_{2\gamma})+gK_{\rho\beta\gamma}^{2}(h_\beta+\eta_\beta)\eta_\gamma + fM_{\rho\beta}u_{1\beta}+\nu R_{\rho\beta}u_{2\beta}, \tag{43}$$

$$F_{\eta\rho} = G_{\rho\beta}^{j}u_{j\beta}. \tag{44}$$

Summing up Eqs. (36) ~ (38) over all elements and solving the resulted $3\times N$ equations, the $3\times N$ unknowns of $u_{1\alpha}^{n+\frac{1}{2}}$, $u_{2\alpha}^{n+\frac{1}{2}}$ and $\eta_{\alpha}^{n+\frac{1}{2}}$ ($\alpha = 1\sim N$) can be resolved from the $3\times N$ known values of $u_{1\alpha}^{n}$, $u_{2\alpha}^{n}$ and η_{α}^{n} ($\alpha = 1\sim N$). Next, summing up Eqs. (39)~(41) over all elements and solving the resulted $3\times N$ equations, the $3\times N$ unknowns of $u_{1\alpha}^{n+1}$, $u_{2\alpha}^{n+1}$ and η_{α}^{n+1} ($\alpha = 1 \sim N$) can be resolved from $6\times N$ known values of $u_{1\alpha}^{n}$, $u_{2\alpha}^{n}$, η_{α}^{n}, $u_{1\alpha}^{n+\frac{1}{2}}$, $u_{2\alpha}^{n+\frac{1}{2}}$ and $\eta_{\alpha}^{n+\frac{1}{2}}$ ($\alpha = 1\sim N$). Then we can proceed to the next step.

4. Results

The division of Kagoshima bay into two-dimensional simplex elements is given in Fig. 1, which is the same as was used in the previous paper[1]. At first some discussions about the stability of the calculation are inevitable. Gray and Lynch[8] investigated the stability conditions for the one-dimensional tidal flow equations without nonlinear convective terms and found that the time step Δt is limited by

$$\Delta t \leqq \frac{1}{4}\frac{(\Delta x_{\min})^2}{\nu}, \tag{45}$$

$$\Delta t \leqq \frac{1}{\sqrt{3}}\frac{\Delta x_{\min}}{\sqrt{gh}}, \tag{46}$$

in the case of the usual Galerkin's finite element method with the two-step Lax-Wendroff time differentiation scheme. Nishi-Sakurajima channel gives the minimum value of Δx to be $\Delta x_{min} \fallingdotseq 10^3$ m. Then taking into account the experimental value of the kinetic viscousity ν being $10^5 \sim 10^7$ (cm^2/sec), Eq. (46) is found to give a more severe restriction for the time step than Eq. (45). If the depth of Nishi-Sakurajima channel is taken to be 40 m, the upper bound for the time step is estimated from Eq. (46) to be about 29 seconds. No other places give more severe restriction to Δt than 29 seconds. Actually, however, the unstable divergence occurs even if Δt is chosen to be 15 seconds. The more rigorous restiction than Eq. (46) might be caused by the two-dimensionality of our problem and the inclusion of the nonlinear convective terms. Calculation is performed by taking 10 seconds for the time step, which gives stable results.

The non-slip boundary condition is adopted at the coast and at the entrance of Kagoshima bay, the harmonic oscillation of water elevation is given with amplitude 1 m and period 12.5 hours. Owing to the oscillatory boundary condition at the open boundary, the water mass transport integrated over one tidal period (WMT) at any cross section of the bay must be vanished, or in other words, the inflow part of WMT (WMTIN) and that of outflow part (WMTOUT) have to be equal to each other. Calculation is performed over five periods. The time variations of WMTIN and WMTOUT at the open boundary and at the center of the bay shown in Fig. 1 are represented in Fig. 2. At the fifth period,

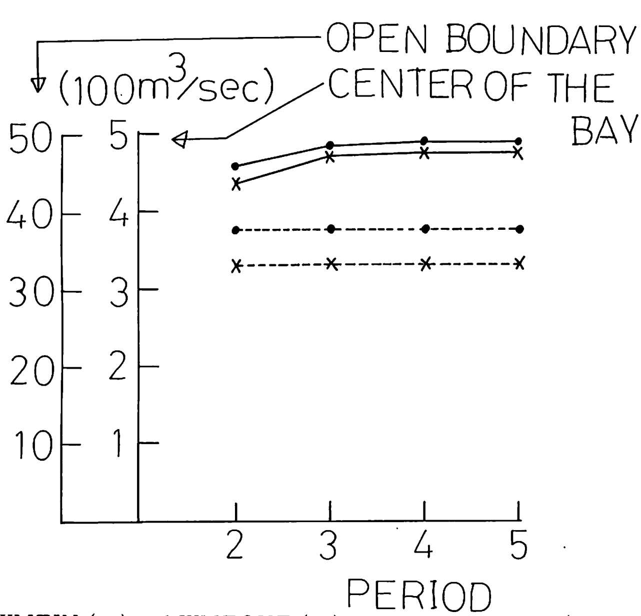


Fig. 2 WMTIN (●) and WMTOUT (×) at the open boundary (dashed line) and at the center of the bay (solid line) of Kagoshima bay with respect to period calculated by EWM.

$$\mathrm{WMTIN} = 3.768 \times 10^3\ \mathrm{m^3/sec}, \qquad \mathrm{WMTOUT} = 3.338 \times 10^3\ \mathrm{m^3/sec},$$
$$\frac{\mathrm{WMTOUT}}{\mathrm{WMTIN}} = 0.886, \tag{47}$$

at the open boundary and

$$\mathrm{WMTIN} = 4.933 \times 10^2\ \mathrm{m^3/sec}, \qquad \mathrm{WMTOUT} = 4.746 \times 10^2\ \mathrm{m^3/sec},$$
$$\frac{\mathrm{WMTOUT}}{\mathrm{WMTIN}} = 0.962, \tag{48}$$

at the center of the bay. The 3.8% loss of water mass at the center of the bay could be admitted from the point of relatively rough division of the bay. The 11.4% loss of water mass at the open boundary could be improved if the finer divisions of there and of the Nishi-Sakurajima channel are employed. In fact, in the case of Shibushi bay [5] where nine nodal points exist at the open boundary, the loss of water mass is estimated to be only 2.8%. In the Nishi-Sakurajima channel, only two nodal points are there and WMTIN is much larger than WMTOUT, which might be also one of the causes of the loss of water mass at the open boundary.

The resulted tidal residual mass transport at fifth period is shown in Fig. 3. As can be seen from the figure, there exists a large anti-clockwise vortex at the center of the bay and water mass inflows from the east side and outflows from the west side of the mouth of the bay. These two features of the tidal residual mass transport are similar to the observational results so far performed. At Nishi-Sakurajima channel, water mass looks like to inflow from the west side and to outflow from the east side of the

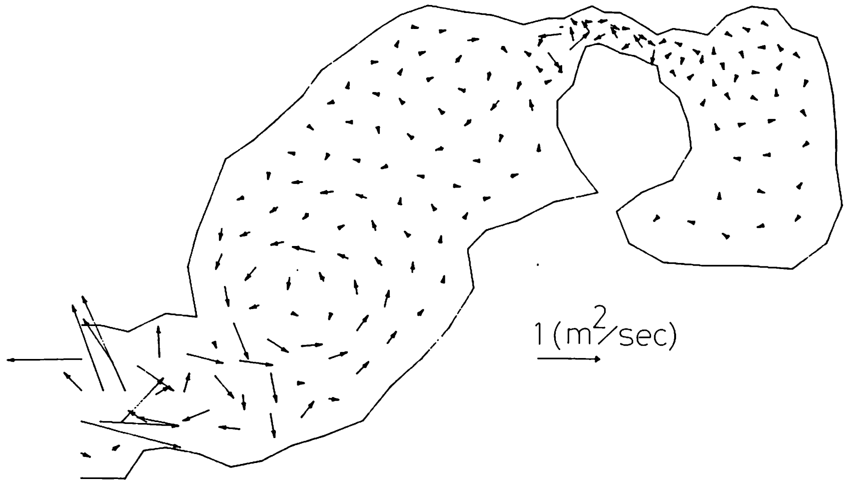


Fig. 3 The distribution of the tidal residual mass transport in Kagoshima bay calculated by EWM.

channel. The water mass transport of there, however, is not well conserved and further investigation by a finer division of the channel must be inevitable. The distribution of the tidal residual velocity is given in Fig. 4, the shape of which is very similar to Fig.3 except the relative smallness of the strength of the anti-clockwise vortex. The tidal residual mass transport consists of that proportional to the tidal

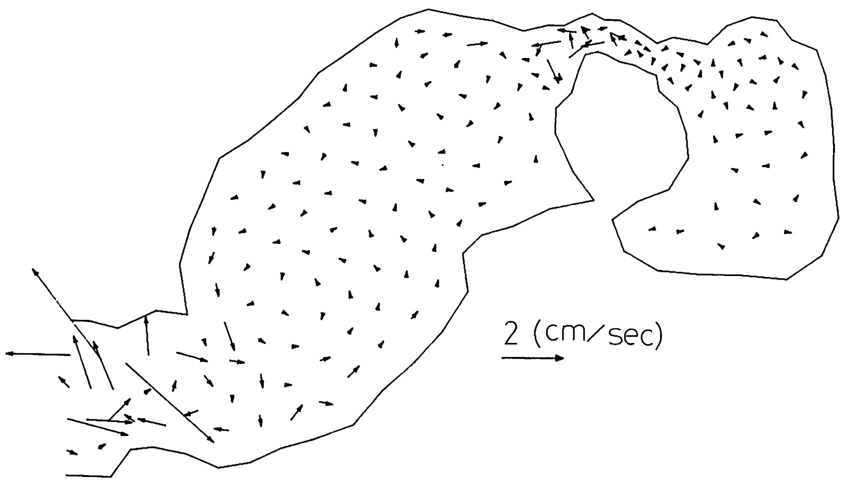


Fig. 4 The distribution of the tidal residual velocity in Kagoshima bay calculated by EWM.

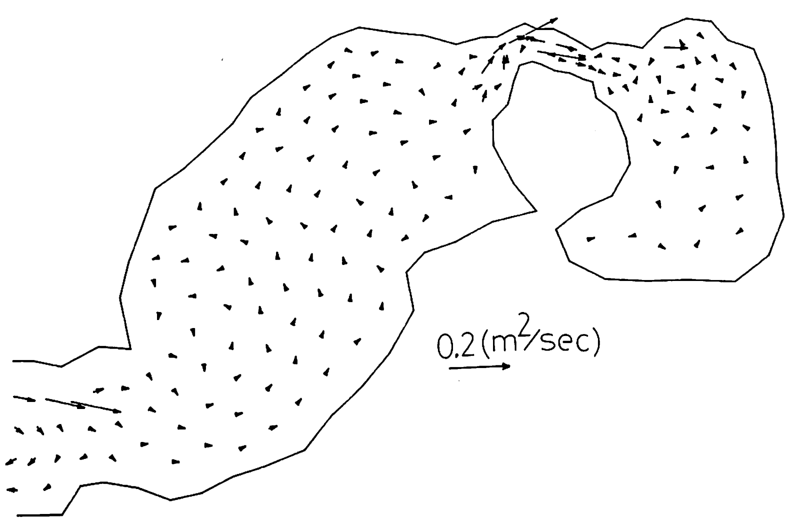
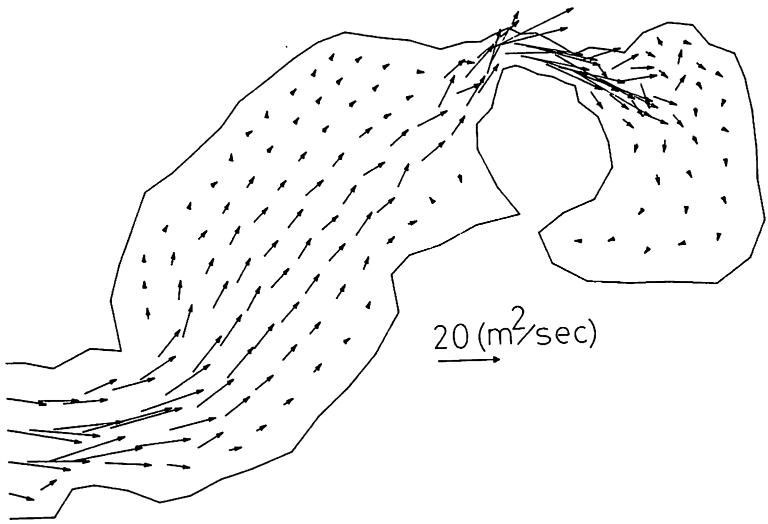
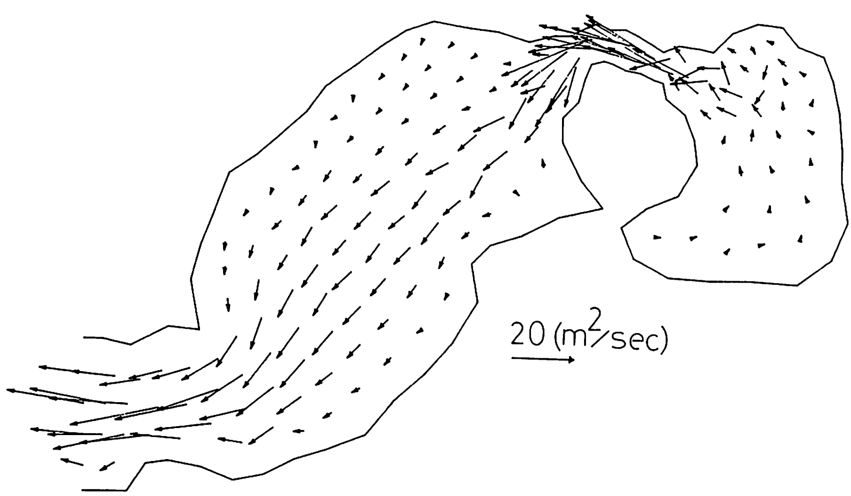


Fig. 5 The distribution of (tidal residual mass transport − tidal residual velocity × depth) in Kagoshima bay calculated by EWM.

residual velocity and of the Stokes mass transport. The Stokes mass transport vanishes if the tidal wave is the standing wave. (Tidal residual mass transport − tidal residual velocity × depth) are depicted in Fig. 5, the main part of which is the Stokes mass transport. The conspicouous differences of this figure from Figs. 3 and 4 are (i)

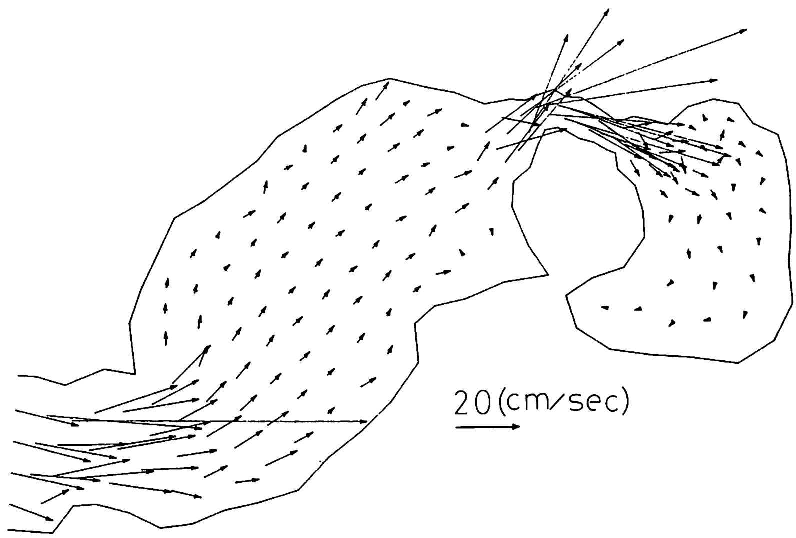


(a)

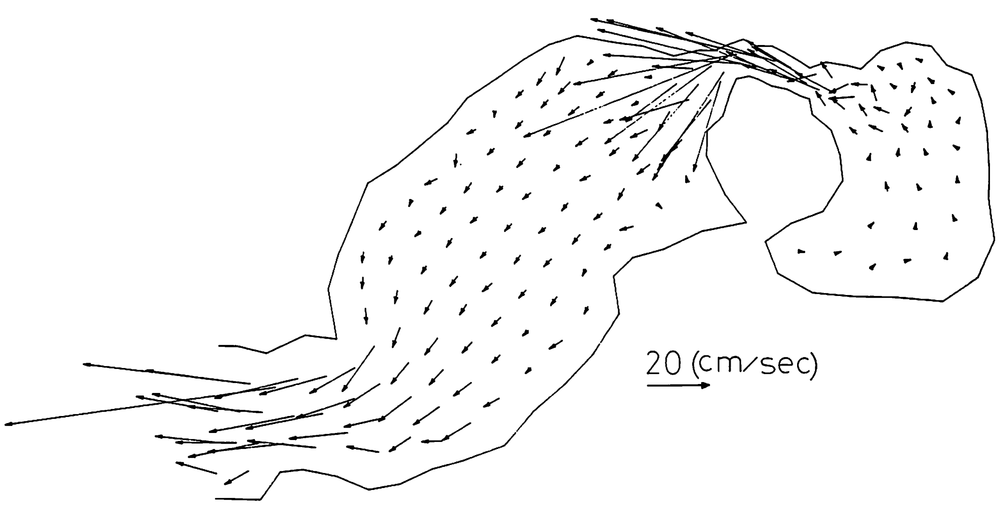


(b)

Fig. 6 Tidal water mass transport distributions at t = 6.25 hour (6a) and at t = 12.5 hour (6b) in Kagoshima bay calculated by EWM.



(a)



(b)

Fig. 7 Tidal velocity distributions at t = 6.25 hour (7a) and at t = 12.5 hour (7b) in Kagoshima bay calculated by EWM.

the water mass inflows from the west side and outflows from the east side of the mouth of the bay and (ii) at the offshore of Tarumizu-city the water mass flows to the southern direction. The distributions of the tidal water mass transport (velocity) at t = 6.25 hour and at t = 12.5 hour are represented in Fig. 6 (Fig. 7).

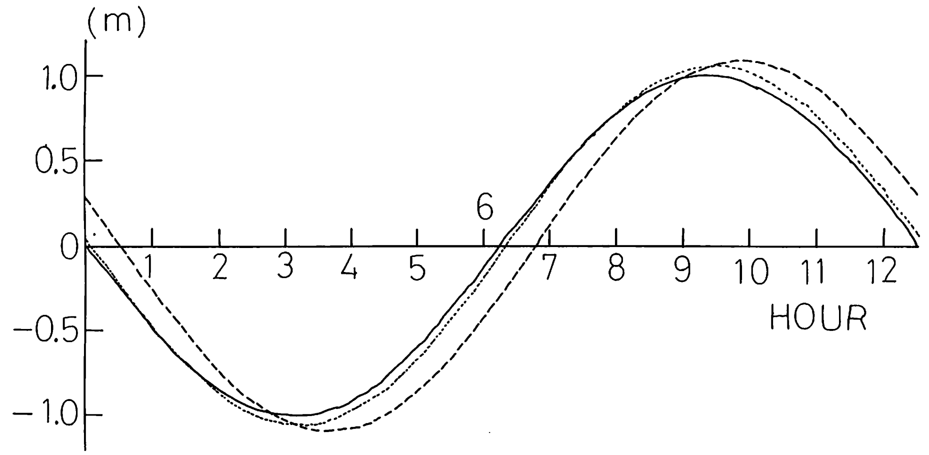


Fig. 8 Time variations of water surface elevations at points A (solid lime), B (dotted lime) and C (dashed lime) shown in Fig. 1 calculated by EWM.

In Fig. 8 is given the time variation of the water surface elevation of the fifth period at the open boundary (A), at off Kagoshima city (B) and at off Fukuyama city (C), where the points A, B and C are shown in Fig. 1. The phase delay of point B (C) with respect to the point A is seen to be about 6 minutes (32 minutes), which was about 20 minutes (90 minutes) in case of the primitive lumped mass matrix method. Although the phase delay seems to be still larger than the observational one, the contradiction between them is much improved in the EWM from the lumped mass matrix case. The amplitude of water surface elevation at the point B (C) is about 5.8% (9.2%) larger than that at the point A, which was about 4.0% (−7.5%) in case of the primitive lumped mass matrix method[1].

5. Discussions about the further improvement

(i) The finer division of the bay is inevitable to improve the water mass conservation, especially at the Nishi-Sakurajima channel and at the open bounndary. The contradiction between the calculated phase delay and the observational one would be decreased also by the finer division.

(ii) An another improvement could be done by adding the outer part of bay into the calculational area. This addition might be especially important to investigate the pattern of tidal residual flow at the mouth of the bay.

(iii) The other boundary conditions, slip boundary condition at the coast, velocity or water mass tranport boundary condition at the open boundary etc., have to be studied

also to examine whether the results owe to the boundary conditions abruptly or not.
(iv) In order to compare the calculated results with the observational ones, it must be inevitable to investigate the influences of winds, to give a more realistic water surface elevation at the open boundary, to extend the method to the three dimensional case and to take into account the influences of density, etc..

Acknowledgements

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Appendix Comparison between EWM and primitive lumped mass matrix method and between conservative form and non-conservative form

For completeness of this paper, EWM is briefly compared with the primitive lumped

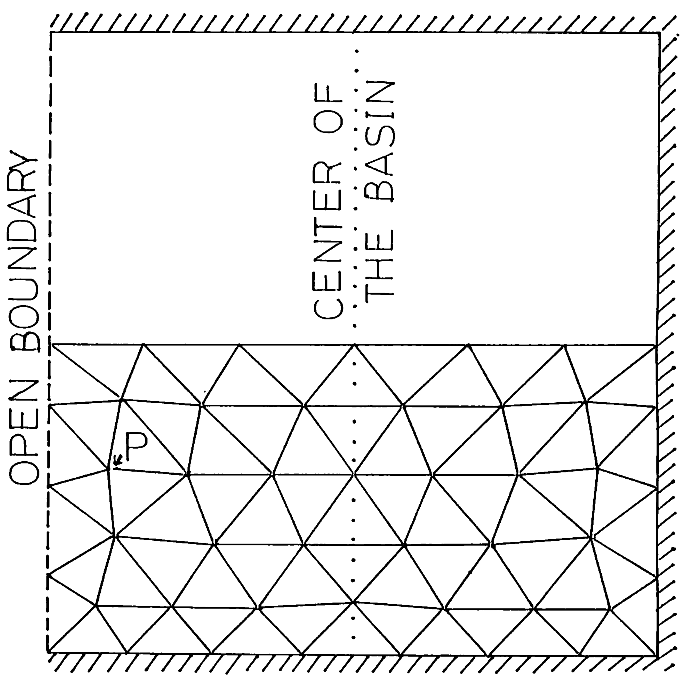

Fig. 9 Model basin and its division into two dimensional simplex elements. Because of the symmetry of the basin, calculations are performed only in the lower half of the basin.

mass matrix method and conservative form of equations with the non-conservative ones. Comparison is made for the rectangular model basin shown in Fig. 9 with constant depth. The non-dimensional governing equations without Coriolis force are given by

$$\frac{\partial v_i}{\partial t}+\varepsilon v_j\frac{\partial v_i}{\partial x_j}+\lambda\frac{\partial \eta}{\partial x_i}-E\varDelta v_i = 0, \tag{A-1}$$

$$\frac{\partial \eta}{\partial t}+\frac{\partial}{\partial x_j}[(1+\varepsilon\eta)v_j] = 0, \tag{A-2}$$

for non-conservative forms and

$$\frac{\partial u_i}{\partial t}+\varepsilon\frac{\partial(u_i v_j)}{\partial x_j}+\lambda(1+\varepsilon\eta)\frac{\partial \eta}{\partial x_i}-E\varDelta u_i = 0, \tag{A-3}$$

$$\frac{\partial \eta}{\partial t}+\frac{\partial u_j}{\partial x_j} = 0, \tag{A-4}$$

for conservative forms. All quantities in Eqs. (A-1)~(A-4) are non-dimensional ones and

$$\varepsilon \equiv \frac{a}{h}, \qquad \lambda \equiv \frac{gT^2h}{l^2}, \qquad E = \frac{\nu T}{l^2}, \tag{A-5}$$

$$u_i = (1+\varepsilon\eta)v_i, \tag{A-6}$$

where a is the amplitude of the tide at the open boundary, T the period of the tide and l denotes representative length of the basin. For EWM, the equations to be summed up over all elements are written in analogy with Eqs. (36)~(41) as

$$\frac{A}{3}q_\alpha^{n+\frac{1}{2}} = \frac{A}{3}q_\alpha^n-\frac{\varDelta t}{2}(V^{-1})_{\alpha\rho}F_{q\rho}^n, \tag{A-7}$$

$$\frac{A}{3}q_\alpha^{n+1} = \frac{A}{3}q_\alpha^n-\varDelta t(V^{-1})_{\alpha\rho}F_{q\rho}^{n+\frac{1}{2}}, \tag{A-8}$$

where q represents v_i $(i = 1\sim2)$ and η for non-conservative case and u_i $(i = 1\sim2)$ and η for conservative case. For the lumped mass matrix method, the equations corresponding to Eqs. (A-7) and (A-8) are

$$\frac{A}{3}q_\alpha^{n+\frac{1}{2}} = M_{\alpha\beta}q_\alpha^n-\frac{\varDelta t}{2}F_{q\alpha}^n, \tag{A-9}$$

$$\frac{A}{3}q_\alpha^{n+\frac{1}{2}} = M_{\alpha\beta}q_\alpha^n-\varDelta tF_{q\alpha}^{n+\frac{1}{2}}, \tag{A-10}$$

where $M_{\alpha\beta}$ is that defined in Eq. (4). In Eqs. (A-7) ~ (A-10), F_q are written as

$$F_{v_i\alpha} = \varepsilon K_{\alpha\beta\gamma}^j v_{j\beta}v_{i\gamma}+\lambda G_{\alpha\beta}^i\eta_\beta+ER_{\alpha\beta}v_{i\beta}, \tag{A-11}$$

$$F_{\eta\alpha} = K_{\alpha\beta\gamma}^j[v_{j\beta}(1+\varepsilon\eta_\gamma)+(1+\varepsilon\eta_\beta)v_{j\gamma}], \tag{A-12}$$

for non-conservative form and

$$F_{u_i\alpha} = \varepsilon K^{j}_{\alpha\beta\gamma}(u_{i\beta}v_{j\gamma} + v_{j\beta}u_{i\gamma}) + \lambda K^{i}_{\alpha\beta\gamma}(1+\varepsilon\eta_\beta)\eta_\gamma + ER_{\alpha\beta}u_{i\beta}, \qquad \text{(A -13)}$$
$$F_{\eta\alpha} = G^{j}_{\alpha\beta}u_{j\beta}, \qquad \text{(A -14)}$$

for conservative form.

Non-dimensional parameters are chosen to be $\varepsilon = 0.05$, $\lambda = 5080$ and $E = 0.01$ corresponding to Yanagi's indoor experiments[9] of $a = 0.5$ (cm), $h = 10$ (cm), T = 360 (sec), $l = 500$ (cm) and $\nu = 7$ (cm^2/sec), which is the fundamental experiment for tidal and tidal residual flows. Water surface elevation is given at the open boundary and non-slip boundary condition is adopted at the coast. Time step Δt is chosen to be 1/3600 taking into account the stability condition of Ref. 8. For EWM calculations are performed over 9 periods in order to get stable results and for primitive lumped mass matrix method only two periods calculations are satisfactory. In fact in the latter case, the results in the third period are equal to those at the second period up to five figures.

The resulted tidal residual mass transports are given in Figs. 10～13 and WMTIN and WMTOUT together with the loss of water mass at the open boundary and at the center of the basin are listed in Table 1. From Figs. 10～13 and Table 1, it can be seen that (i) the water mass is lost completely in the primitive lumped mass matrix case independent of the forms of the employed equations and (ii) in the EWM, the conservative forms of governing equations looks like to be more adequate than the non-conservative ones to estimate the tidal residual flow.

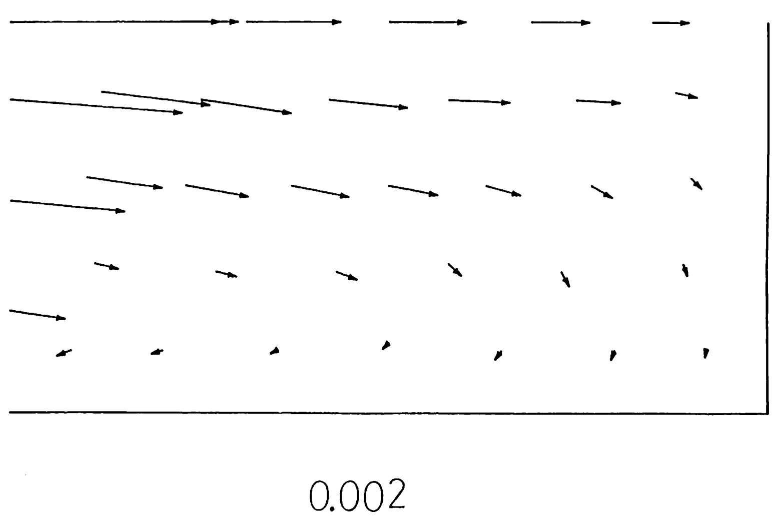

Fig. 10 The distribution of the tidal residual mass transport in a model basin calculated by the primitive lumped mass matrix method with non-conservative form of equations.

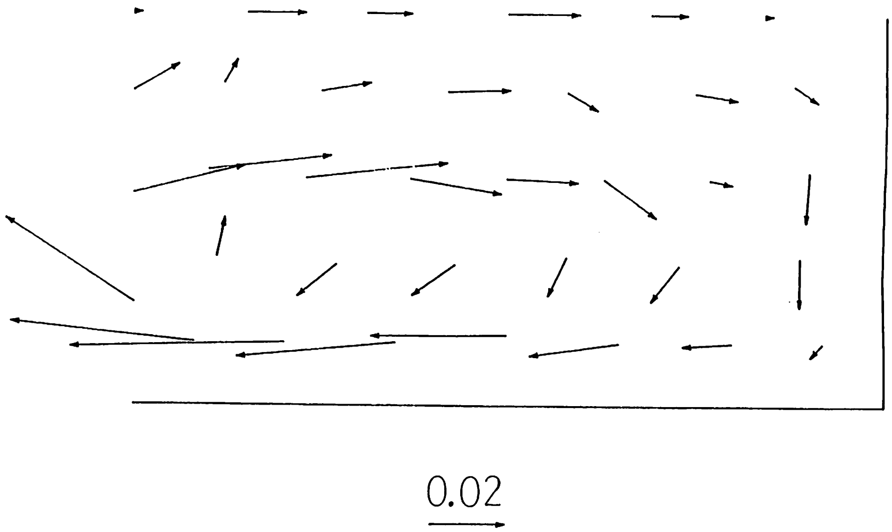

Fig. 11 The distribution of the tidal residual mass transport in a model basin calculated by EWM with non-conservative form of equations.

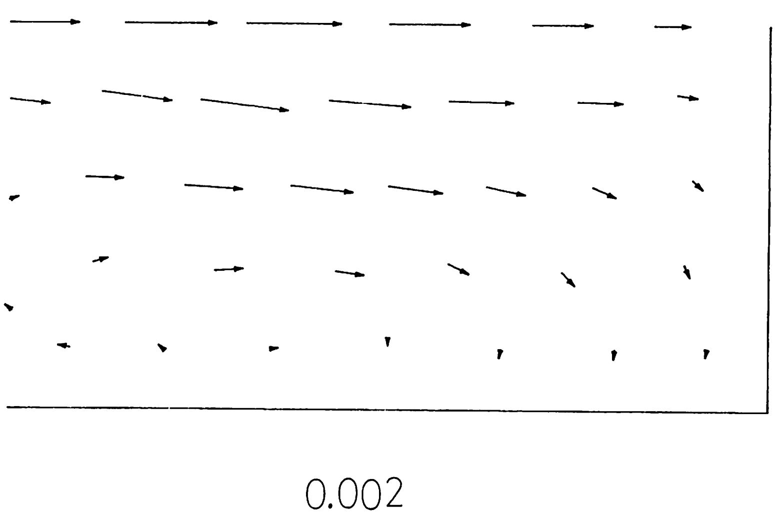

Fig. 12 The distribution of the tidal residual mass transport in a model basin calculated by the primitive lumped mass matrix method with conservative form of equations.

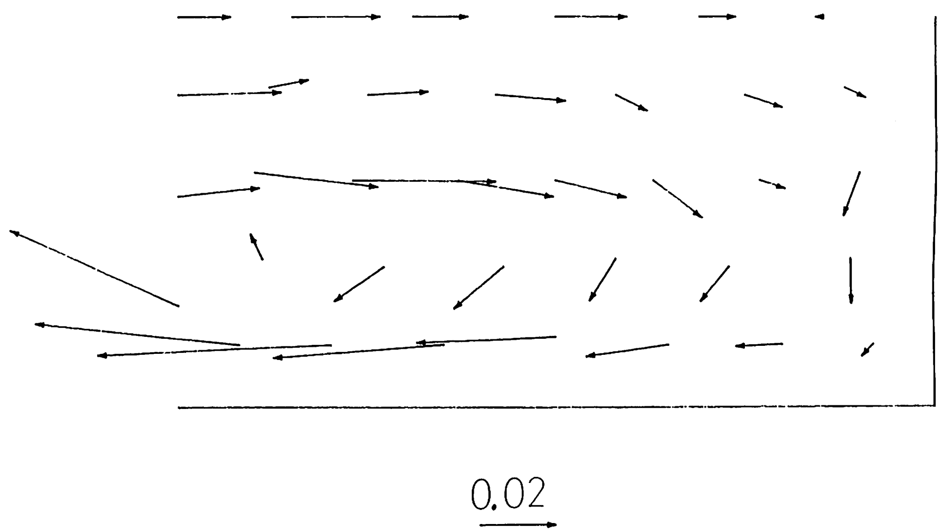

Fig. 13 The distribution of the tidal mass transport in a model basin calculated by EWM with conservative form of equations.

Table 1. Balances of water mass tranport in a model basin calculated by EWM and the primitive lumped mass matrix method with conservative and non-conservative forms of equations.

		Lumped mass matrix method		EWM (ninth period)	
		non-conservative form	conservative form	non-conservative form	conservative form
open boundary	WMTIN	0.277×10^{-2}	0.462×10^{-3}	0.861×10^{-2}	1.126×10^{-2}
	WMTOUT	0	0.018×10^{-3}	0.695×10^{-2}	0.994×10^{-2}
	$\frac{\text{WMTOUT}}{\text{WMTIN}}$	0	0.039	0.807	0.883
	Loss of water mass	100%	96.1%	19.3%	11.7%
center of the basin	WMT IN	0.948×10^{-3}	0.105×10^{-2}	0.795×10^{-2}	0.805×10^{-2}
	WMTOUT	0.019×10^{-3}	0	0.760×10^{-2}	0.799×10^{-2}
	$\frac{\text{WMTOUT}}{\text{WMTIN}}$	0.020	0	0.956	0.992
	Loss of water mass	98.0%	100%	4.4%	0.8%

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