SOME TOPICS IN CONFORMAL FINSLER GEOMETRY

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Preface

The conformal theory in Finsler geometry has been discussed by many author and many results have been obtained. M. Hashiguchi ([Ha]) treated the conformal theory of Finsler metrics and obtained respective conditions under which a Finsler space is conformal to a Berwald space and to a locally Minkowskian space. M. Hashiguchi and Y. Ichijyō ([Ha-Ic]) treated conformal transformations of generalized Berwald space, especially Wagner space.

T. Aikou ([Ai1], [Ai2]) has introduced the notion of locally conformal Berwald manifold, which is also called a Wagner manifold, and investigated the class of Finsler manifolds which are locally conformal to a Berwald manifold using the so-called average Riemannian metric and averaged connection defined in [Ma-Ra-Tr-Ze] and [To-Et], respectively.

The purpose of this thesis is to study the conformal theory in Finsler geometry using the Finsler-Weyl structure which is a natural extension of Weyl structures in Riemannian geometry. In the last chapter we shall characterize conformal flatness of Finsler metrics and Randers metrics.

Chapter 1 is about the preliminaries. First we shall explain Ehresmann connections and non-linear connections on tangent bundles, then we shall introduce a connection usually defined to be a covariant derivation which satisfies the Leibniz rule. In the last section we will discuss conformal classes, Weyl connections and Lyra connections.

Chapter 2 discusses some basic concepts of Finsler manifolds, such as Minkowski norms and Finsler metrics. There are many examples of Finsler manifolds, such as smooth manifold with Riemannian metrics and smooth manifold with Randers metrics, where a Randers metric is a typical non-Riemannian Finsler metric.

Chapter 3 discusses Berwald connections on Finsler manifolds. By a clever observation of Z. I. Szabó , if a Finsler manifold is a Berwald manifold, then its Berwald connection is induced from the Levi-Civita connection on a smooth manifold with a Riemannian metric, and such a Riemannian metric is given by the so-called averaged Riemannian metric obtained from the given Finsler function [Ma-Ra-Tr-Ze]. Landsberg manifolds also form a special class of Finsler manifolds, which includes Berwalds manifolds. Following [To-Et], we shall define the averaged connection obtained from the Berwald connection.

Chapter 4 introduces another Finsler connection which satisfies the so-called almost G-compatibility, which is called the Rund connection. Curvature and torsion of Rund connection is also defined in the next section. We list up some identities concerning curvature and torsion as well.

Chapter 5 investigates geometry of conformal Finsler manifolds. We shall extend the notion of Weyl structures to the category of Finsler geometry, specifically the Finsler-Weyl connection and Wagner connection in Riemannian geometry. In the last section we shall discuss the conformal flatness of Finsler metrics and conformal flatness of Randers metrics.

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Contents

Chapter 1

Connections for tangent bundle

1.1 Ehresmann connections in tangent bundles

Let $\pi : TM \longrightarrow M$ be the tangent bundle over a smooth manifold M of $\dim M = n$ and $d\pi : TTM \longrightarrow TM$ its derivative . Let $y \in T_xM$ be a tangent vector at $x \in M$, where $T_xM = \pi^{-1}(x)$ is the tangent space at $x \in M$. Then the pair $v = (x, y)$ denotes a point in TM. We denote by $\widetilde{TM}$ the pull-back of tangent bundle TM by $\pi : TM \longrightarrow M$:

$$\widetilde{TM} = \{(v, y) \in TM \times TM | y \in T_{\pi(v)}M\} = \coprod_{v \in TM} T_{\pi(v)}M.$$

The following diagram is commutative.

$$\begin{array}{ccc} \widetilde{TM} & \xrightarrow{\ \widetilde{\pi}\ } & TM \\ \downarrow & \circlearrowright & \downarrow \pi \\ TM & \xrightarrow[\ \pi\ ]{} & M \end{array}$$

The fiber $(\widetilde{TM})_v$ over $v \in TM$ is isomorphic to the fiber $T_{\pi(v)}M$ over $\pi(v)$.

A tangent vector $Z_v \in T_vTM$ at $v \in TM$ is said to be *vertical* if $d\pi_v(Z_v) = 0$ is satisfied. Any integral curve of vertical vector fields lies entirely in the fiber $T_{\pi(v)}M$. We denote by V_v the tangent space of all vertical tangent vector at $v \in TM$ which is identified with the tangent space $T_v(T_{\pi(v)}M)$ at v in the fiber $T_{\pi(v)}M$. Since π is a submersion, the tangent space T_vTM at $v \in TM$ is mapped onto the tangent space $T_{\pi(v)}M$ at $\pi(v) \in M$ by the derivative $d\pi_v$:

$$V_v = T_v(T_{\pi(v)}M) = \ker\{d\pi_v : T_vTM \longrightarrow T_{\pi(v)}M\}.$$

Then we define a space V by

$$V \stackrel{\text{def}}{=} \coprod_{v \in TM} V_v = \coprod_{v \in TM} T_v(T_{\pi(v)}M) = \ker\{\widetilde{d\pi} : TTM \longrightarrow \widetilde{TM}\}$$

and a map $p : V \to TM$ by $p(V_v) = v$. Then $p : V \to TM$ has a bundle structure over TM of rank(V)=n. The bundle V is called the *vertical sub-bundle* of TTM. The bundle V is always integrable, that is, the space of sections of V is a Lie algebra under the usual Lie bracket of tangent bundle. Since the quotient bundle TTM/V is isomorphic to the pull-back bundle $\widetilde{TM}$, we obtain the following short exact sequence of tangent bundle over TM:

$$\mathbb{O} \longrightarrow V \xrightarrow{\ \iota\ } TTM \xrightarrow{\ \widetilde{d\pi}\ } \widetilde{TM} \longrightarrow \mathbb{O}, \tag{1.1}$$

where $\iota : V \hookrightarrow TTM$ is the inclusion and $\widetilde{d\pi} = (\pi, d\pi)$, which implies the natural identification $TTM \cong V \oplus \widetilde{TM}$.

Since any point $(\pi(v), Z) \in T_{\pi(v)}M$ is naturally identified with the velocity vector

$$\left.\frac{dc}{dt}\right|_{t=0} \in T_v(T_{\pi(v)}M)$$

of a curve $c(t) = v + (\pi(v), tZ)$ in the fiber $T_{\pi(v)}M$. Therefore the induced bundle $\widetilde{TM}$ is isomorphic to the sub-bundle V:

$$\widetilde{TM} \cong V \tag{1.2}$$

We shall denote by $m_\bullet : \mathbb{R}^+ \times TM \longrightarrow TM$ the natural action of the multiplier group $\mathbb{R}^+$ by scalar multiplication: $m_\lambda(v) = \lambda \cdot v$ for any $v \in TM$ and $\lambda \in \mathbb{R}^+$. This action m of $\mathbb{R}^+$ on TM induces a vector field $\mathcal{E}$ along the fibers by

$$\mathcal{E}_v(f) := \left.\frac{d}{d\lambda}\right|_{\lambda=0} f(e^\lambda \cdot v) \tag{1.3}$$

for all $f \in C^\infty(TM)$, that is, $\mathcal{E}_v = (v, v)$. This field $\mathcal{E}$ is called the *Liouville vector field* in TM, or as a section of V, called the *tautological section* of V.

Since every subspace of T_vTM complementary to V_v is mapped isomorphically onto the tangent space $T_{\pi(v)}M$, there is no canonical subspace complementary to V_v. Thus we shall fix a selection of complementary subspace at each point $v \in TM$.

Definition 1.1. Let $\pi : TM \longrightarrow M$ be the tangent bundle over M. An *Ehresmann connection* on π is a collection $H = \{H_v | v \in TM\}$ of subspace $H_v \subset T_vTM$ such that

1. The assignment $TM \ni v \longmapsto H_v \subset T_vTM$ depends on $v \in TM$ smoothly,
2. $d\pi_v : H_v \longrightarrow T_{\pi(v)}M$ is a linear isomorphism for all $v \in TM$ and H_v is complementary to V_v:
$$T_vTM = V_v \oplus H_v. \tag{1.4}$$

The subspace H_v is called a *horizontal subspace* at $v \in TM$.

An alternative method to specify an Ehresmann connection is to give a splitting $\theta : TTM \longrightarrow V$ of the exact sequence (1.1)

$$\mathbb{O} \longrightarrow V \underset{\theta}{\overset{\iota}{\rightleftarrows}} TTM \xrightarrow{\widetilde{d\pi}} \widetilde{TM} \longrightarrow \mathbb{O},$$

namely $\theta : TTM \longrightarrow V$ is a bundle morphism satisfying $\theta \circ \iota = \mathrm{Id}_V$ for the identity morphism Id_V of V. Therefore θ is a projection from TTM onto V. We set

$$H_v \overset{\text{def}}{=} \ker(\theta_v) \tag{1.5}$$

at each point $v \in TM$. Then $T_vTM/\ker(\theta_v) \cong im(\theta_v) = V_v$. Therefore we obtain the splitting (1.4). Defining
$$H \overset{\text{def}}{=} \coprod_{v \in TM} H_v, \tag{1.6}$$

we obtain a bundle $p : H \to TM$ by $p(H_v) = v$, which is also isomorphic to the induced bundle $\widetilde{TM}$. Then we have the direct-sum decomposition

$$TTM = V \oplus H \tag{1.7}$$

Definition 1.2. The bundle $H = \ker(\theta)$ is called the *horizontal sub-bundle* defined by θ. A section of H is called a *horizontal vector field* on TM

An Ehresmann connection for TM is a subbundle $H \subset TTM$ complementary to V.

Definition 1.3. An Ehresmann connection θ of a tangent bundle $\pi : TM \mapsto M$ is called a *non-linear connection* for TM if it satisfies the following conditions:

(N-1) The distribution $H : TM \ni v \longmapsto H_v \subset T_vTM$ is smooth on $TM \setminus \{0_M\}$ and is continuous on the whole of TM, where 0_M is the zero section of TM.

(N-2) The distribution H is invariant under the action m of $\mathbb{R}^+$ on TM, i.e.,

$$dm_\lambda(H_v) = H_{m_\lambda(v)} \tag{1.8}$$

for any $\lambda \in \mathbb{R}^+$ and $v = (x, y) \in TM$.

Remark 1.1. If H is smooth on the whole of TM, then it is called *linear*.

Usually an Ehresmann connection θ of a tangent bundle is assumed to be smooth on the whole total space TM. However, we assume the smoothness of θ only on the outside of the zero-section for application to Finsler geometry.

Let θ be a non-linear connection for TM. A vector field X in M is parallel along a regular curve $c : [a, b] \to M$ with respect to θ if it satisfies the ordinary differential equation

$$(X \circ c)^* \theta = 0, \tag{1.9}$$

or, equivalently, its velocity vector field $(X \circ c)'$ is always horizontal, i.e., $(X \circ c)'(t) \in H_{(X \circ c)(t)}$ for all $t \in [a, b]$. Equation (1.9) has a unique solution X_v for each initial value $v \in T_{c(a)}M$, on which it depends smoothly. The parallel transport $P_{c(t)} : T_{c(a)}M \to T_{c(t)}M$ defined by

$$P_{c(t)}(v) = X_v(t) \tag{1.10}$$

has the homogeneity property

$$P_{c(t)}(\lambda \cdot v) = \lambda \cdot P_{c(t)}(v) \tag{1.11}$$

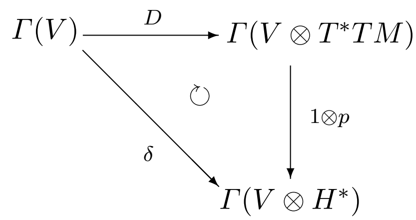
for any $v \in T_{c(a)}M$ and $\lambda \in \mathbb{R}^+$, where we write $\lambda \cdot v := m_\lambda(v)$ for simplicity.

Proposition 1.1. *An Ehresmann connection θ is a linear connection if and only if the parallel translation P_c along any curve $c = c(t)$ in M is a linear isomorphism between the fibers.*

1.2 Connections associated with non-linear connection

Let A^k be the space of smooth k-form, in particular A^0 is the space of smooth function, $\Gamma(F)$ be the space of smooth sections of a vector bundle F and $A^k(F)$ is the space of smooth k-forms with values in F. Then $A^0(F) = \Gamma(F)$.

If a non-linear connection θ is specified in TM, then there exists a *partial connection* $\delta : \Gamma(V) \to \Gamma(V \otimes H^*)$ along $H = ker(\theta)$ in the bundle V, where H^* is the dual bundle of H. Moreover, any partial connection δ can be extended to a connection $D : \Gamma(V) \to \Gamma(V \otimes T^*TM) = A^1(V)$ so that the following diagram is commutative ([Ba-Bo]):



where $p : T^*TM \to H^*$ is the natural projection and T^*TM is the dual bundle of TTM.

Suppose that a non-linear connection θ is given in TM. Since we do not assume the differentiability of θ at the zero-section, the parallel translation P_c along any curve c in M is compatible only with scalar multiplication. Therefore we cannot define a connection ∇ on TM from any non-linear connection θ in general. However, we shall show that any θ induces a connection D on the vertical subbundle V as the extension of a partial connection δ.

A *connection* in the bundle V is usually defined to be a *covariant derivation* in V, i.e., as a homomorphism $D : \varGamma(V) \to A^1(V)$ satisfying the *Leibniz rule*. We shall introduce a connection D associated with a non-linear connection θ.

Since the vertical sub-bundle V is isomorphic to the induced bundle $\widetilde{TM}$, any vector field X in M is naturally lifted to a section $X^V \in \varGamma(V)$. The section X^V is defined as a vector field which is tangent to the curve $c(t) = (x, y + tX(x))$ in the fiber T_xM at $t = 0$. The map $T_xM \ni X(x) \longmapsto X^V(v) \in V_v$ is an isomorphism. Thus the vector field X^V is uniquely determined by X. So the vector field X^V is called the *vertical lift* of X. In the sequel we use the superscript V for the vertical lifts of vector fields on M.

On the other hand, for any vector field X in M, there exists a unique section X^H of H such that $d\pi_v(X^H) = X_{\pi(v)}$ at any point $v \in TM$. The vector field X^H on the total space TM is called the *horizontal lift* of X. In the sequel we use the superscript H for the horizontal lifts of vector fields on M

In this thesis we use extensively the coordinate system $\{\pi^{-1}(U), (x^i, y^i)_{1\le i\le n}\}$ in TM induced from a coordinate system $\{U, (x^i)\}_{1\le i\le n}$ in M, where $y^1, \cdots, y^n$ are the fibre coordinates in each T_xM, $x \in U$. Then the vertical lift X^V of $X = \sum X^i(\partial/\partial x^i)$ is given by

$$X^V = \sum X^i \frac{\partial}{\partial y^i}, \tag{1.12}$$

and the horizontal lift X^H of X is given by

$$X^H = \sum X^i \left(\frac{\partial}{\partial x^i} - \sum N_i^l \frac{\partial}{\partial y^l} \right), \tag{1.13}$$

for some functions N_j^i defined in $\pi^{-1}(U)$. These local function N_j^i are called the *coefficients* of

H. The coefficients N^i_j are smooth away from the zero-section. The assumption (N-2) means that the coefficients N^i_j are homogeneous of degree one with respect to the variables $y^1, \cdots, y^n$. Thus, by Euler's theorem,

$$\sum \frac{\partial N^i_j}{\partial y^l} y^l = N^i_j. \tag{1.14}$$

The Liouville vector field is given by

$$\mathcal{E} = \sum y^l \frac{\partial}{\partial y^l}. \tag{1.15}$$

From the homogeneity condition (N-2), the coefficients N^i_j are linear in the variables $y^1, \cdots, y^n$ if H is linear. Thus, if H is linear, the coefficients N^i_j are written as

$$N^i_k = \sum y^j \Gamma^i_{jk},$$

where $\Gamma^i_{jk} = \Gamma^i_{jk}(x)$ are coefficients of affine connection in TM.

Since any vector field Y on M is a smooth map $Y : M \to TM$ such that $\pi \circ Y = id$, its derivative $dY_x : T_xM \to T_{Y(x)}TM$ satisfies

$$d\pi \left(dY\left(\frac{dc}{dt}\right) - \left(\frac{dc}{dt}\right)^H \right) = 0$$

for any reguler curve c in M. Then the equation

$$dY\left(\frac{dc}{dt}\right) = \mathcal{L}_{(dc/dt)^H} Y^V + \left(\frac{dc}{dt}\right)^H$$

holds, where $\mathcal{L}_{X^H}$ denotes the Lie derivative by X^H. Since $H = ker(\theta)$, we obtain

$$(Y \circ c)^*\theta\left(\frac{d}{dt}\right) = \theta\left(\mathcal{L}_{(dc/dt)^H} Y^V + \left(\frac{dc}{dt}\right)^H\right) = \theta(\mathcal{L}_{(dc/dt)^H} Y^V),$$

and Y is parallel with respect to θ if and only if

$$\theta(\mathcal{L}_{X^H} Y^V) = 0$$

for all $X \in \Gamma(TM)$. Thus it is natural to define $D : \Gamma(V) \to \Gamma(V \otimes H^*)$ by

$$D_{X^H} Y^V := \theta(\mathcal{L}_{X^H} Y^V) = [X^H, Y^V]. \tag{1.16}$$

Let $\{U\}$ be an open cover of M. A vector bundle is said to be *relatively flat* if the transition maps G_{UV} of V depend only on base point $x \in M$. The family G_{UV} is given by $G_{UV} = \pi^* g_{UV}$ where $\{g_{UV}\}$ are transition maps. Since the vertical sub-bundle V is relatively flat, we can define D so that D is flat in the vertical direction, i.e.,

$$D_{X^V} Y^V = 0. \tag{1.17}$$

Definition 1.4. The connection $D : \Gamma(V) \to \Gamma(V \otimes T^*TM) := A^1(V)$ defined by (1.16) and (1.17) is called the canonical connection on V associated with the given non-linear connection θ.

From (1.15) and (1.17), we have $D_{X^V}\mathcal{E} = X^V$, and the homogeneity condition (1.8) implies $D_{X^H}\mathcal{E} = \mathcal{L}_{X^H}\mathcal{E} = 0$. Therefore the given non-linear connection θ is recovered by D.

Proposition 1.2. *The canonical connection D associated with θ satisfies*

$$D\mathcal{E} = \theta \tag{1.18}$$

for the tautological section $\mathcal{E}$ of V.

1.3 Levi-Civita connection of Riemannian manifolds

A Riemannian metric g on a smooth manifold M is a smooth assignment $g : M \ni x \mapsto g_x$, of an inner product $g_x : T_xM \times T_xM \to \mathbb{R}$ on T_xM. Any Riemannian metric g induces a Riemannian metric $\widetilde{g}$ on V by defining

$$\widetilde{g}(X^V, Y^V) = g(X, Y). \tag{1.19}$$

Definition 1.5. A linear connection ∇ is said to be *metrical* if its parallel transport $P_c : TM \to TM$ with respect to ∇ along any curve $c = c(t)$ in M preserves the metric $\widetilde{g}$.

Let H be the linear Ehresmann connection defined by ∇. Let X be a vector field in M, and let $\{\varphi_t\}$ be the one-parameter group of local transformations generated by X. Denoting by $\{\varphi_t^H\}$ the horizontal lift of $\{\varphi_t\}$ with respect to ∇, we obtain

$$\varphi_t^{H*}\widetilde{g} = \widetilde{g}$$

or equivalently

$$\mathcal{L}_{X^H}\widetilde{g} = 0$$

where X^H is the horizontal lift of X with respect to ∇. The Lie derivative $\mathcal{L}_{X^H}\widetilde{g}$ is given by

$$\begin{aligned}(\mathcal{L}_{X^H}\widetilde{g})(Y^V,Z^V) &= X^H\widetilde{g}(Y^V,Z^V) - \widetilde{g}(\mathcal{L}_{X^H}Y^V,Z^V) - \widetilde{g}(Y^V,\mathcal{L}_{X^H}Z^V)\\ &= X^H\widetilde{g}(Y^V,Z^V) - \widetilde{g}((\nabla_X Y)^V,Z^V) - \widetilde{g}(Y^V,(\nabla_X Z)^V)\\ &= Xg(Y,Z) - g(\nabla_X Y,Z) - g(Y,\nabla_X Z)\\ &:= (\nabla_X g)(Y,Z).\end{aligned}$$

Then we have

Proposition 1.3. *In a Riemannian manifold (M,g) there exists a unique linear connection ∇^g of TM such that*

(1) ∇^g *is metrical:*

$$\nabla^g g = 0 \tag{1.20}$$

(2) ∇^g *is torsion free:*

$$\nabla^g_X Y - \nabla^g_Y X - [X,Y] = 0 \tag{1.21}$$

for all $X,Y \in \Gamma(TM)$.

Definition 1.6. The linear connection ∇^g is called the *Levi-Civita connection* of (M,g).

We suppose that the curvature $R^{\nabla^g} = \nabla^g \circ \nabla^g$ of the Levi-Civita connection ∇^g vanishes identically. Then $R^{\nabla^g} = 0$ is an integrability condition for the system of differential equations

$$dA = -\omega^g A,$$

where ω^g is the connection form of ∇^g with respect to the natural frame field $\left\{\frac{\partial}{\partial x^1},\cdots,\frac{\partial}{\partial x^n}\right\}$ on U and $A = (A^i_j)$ is a $GL(n,\mathbb{R})$-valued smooth function on U. Since ∇^g is torsion-free, we have $\omega^g \wedge dx = 0$ and

$$\frac{\partial A^i_j}{\partial x^k} = \frac{\partial A^i_k}{\partial x^j}$$

in local coordinates. Therefore there exist some local functions $f^i(x^1,\cdots,x^n)$ such that $A^i_j = \frac{\partial f^i}{\partial x^j}$. Then, if we take a change of local coordinate as $\overline{x}^i = f^i(x^1,\cdots,x^n)$, the connection form $\overline{\omega}^g$ of R^{∇^g} with respect to $(\overline{x},\overline{y})$ vanishes on U. Hence the components $\overline{g}_{ij}$ of g with respect to $\left\{\frac{\partial}{\partial \overline{x}^1},\cdots,\frac{\partial}{\partial \overline{x}^n}\right\}$ are constans. Since $(\overline{g}_{ij})$ is a positive-definite matrix, we may assume that $\overline{g}_{ij} = \delta_{ij}$ in such a local coordinate system $(\overline{x}^1,\cdots,\overline{x}^n)$, namely (M,g) is *locally Euclidean.*

Theorem 1.1. *The Levi-Civita connections ∇^g is flat if and only if (M,g) is locally Euclidean.*

1.4 Conformal class, Weyl connections and Lyra connections

Let ∇ be a linear connection on TM of a Riemannian manifold (M, g). We suppose that parallel translation P_c along any curve $c = c(t)$ in M is always a conformal map between tangent spaces, namely, P_c preserves the angle of two vector fields on M.

Let H be the linear Ehresmann connection determined by the given ∇, and let $\widetilde{g}$ be the metric on V induced from the given g. Then the assumption above means that the induced metric $\widetilde{g}$ on V is preserved up to a conformal factor by the parallel translations with respect to H. For any vector field X on M and the one parameter group φ_t of local transformations generated by X, this assumption is given by

$$\varphi_t^{H*}\widetilde{g} = \exp\left(2\int_{\varphi_t} w_g\right)\widetilde{g} \tag{1.22}$$

for some one-form $w_g \in A^1(M) := \Gamma(T^*M)$, where φ_t^{H*} is the horizontal lift of φ_t with respect to H. Thus the Lie derivative $\mathcal{L}_{X^H}\widetilde{g}$ is given by

$$\mathcal{L}_{X^H}\widetilde{g} = 2w_g(X)\widetilde{g}, \tag{1.23}$$

where the horizontal lift X^H of $X \in \Gamma(TM)$ is defined with respect to H.

We have

$$\mathcal{L}_{X^H}Y^V = (\nabla_X Y)^V \tag{1.24}$$

and this implies

$$(\mathcal{L}_{X^H}\widetilde{g})(Y^V, Z^V) = (\nabla_X g)(Y, Z)$$

for all $X, Y \in \Gamma(TM)$. Consequently the assumption (1.23) can be written as

$$\nabla g = 2w_g \otimes g \tag{1.25}$$

Two Riemannian metrics g and $\overline{g}$ on M are said to be conformally equivalent if there exists a smooth function σ on M such that $\overline{g} = e^{2\sigma(x)}g$. This definition induces an equivalence relation on Riemannian metrics on M and the equivalence class of g is called the conformal class of g and denoted by $\mathcal{C}$:

$$\mathcal{C} = \left\{ e^{2\sigma(x)}g \,\middle|\, \sigma \in A^0 := C^\infty(M) \right\}.$$

For any metric $\overline{g} = e^{2\sigma(x)}g$ in the class $\mathcal{C}$, we have

$$\nabla\overline{g} = 2e^{2\sigma(x)}(d\sigma + w_g) \otimes g = 2(d\sigma + w_g)\overline{g}.$$

Thus, setting

$$w_{\overline{g}} = w_g + d\sigma, \tag{1.26}$$

we obtain $\nabla \overline{g} = 2w_{\overline{g}} \otimes \overline{g}$, and thus the parallel displacement P_c with respect to ∇ is also a conformal map with respect to the metric $\overline{g} \in \mathcal{C}$. This shows that ∇ preserves the conformal class $\mathcal{C}$ if and only if there exists a map $w : \mathcal{C} \ni g \longmapsto w_g \in A^1$ satisfying (1.25) and (1.26).

Definition 1.7. The pair $(\mathcal{C}, w)$ of a conformal class $\mathcal{C}$ and a map $w : \mathcal{C} \ni g \longmapsto w_g$ satisfying (1.26) is called a *Weyl structure* on M. A symmetric linear connection ∇ on M is called a *Weyl connection* of $(\mathcal{C}, w)$ if ∇ preserves the conformal class $\mathcal{C}$, that is, ∇ satisfies (1.25) for all $g \in \mathcal{C}$. The 1-form w_g corresponding to $g \in \mathcal{C}$ is called a *Lee form* of $(\mathcal{C}, w)$.

The Weyl connection of $(\mathcal{C}, w)$ is thus torsion free but not metric preserving. On the other hand, there exists a unique connection $\overline{\nabla}$ such that $\overline{\nabla}$ is metrical with respect to g:

$$\overline{\nabla} g = 0$$

and $\overline{\nabla}$ is semi-symmetric:

$$\overline{\nabla}_X Y - \overline{\nabla}_Y X - [X, Y] = w_g(Y)X - w_g(X)Y.$$

Such a connection $\overline{\nabla}$ is called the *Lyra connection* of $(\mathcal{C}, w)$([Se-Va]). The relation between ∇ and $\overline{\nabla}$ is given by

$$\nabla_X Y = \overline{\nabla}_X Y - w_g(X)Y.$$

Chapter 2

Finsler manifolds

2.1 Minkowski norms in vector space

Let $\mathbb{V}$ be a vector space of dim $\mathbb{V} = n$

Definition 2.1. A function $L : \mathbb{V} \longrightarrow \mathbb{R}$ is called a *Minkowski norm* if the following conditions are satisfied.

(1) For every $v \in \mathbb{V}, L(v) \geqslant 0$ and the equality holds if and only if $v = 0$

(2) For every $v \in \mathbb{V}$ and every $\lambda > 0$, the homogeneity condition

$$L(\lambda v) = \lambda L(v) \tag{2.1}$$

is satisfied.

(3) For all $v, w \in \mathbb{V}$, the triangle inequality

$$L(v + w) \leqslant L(v) + L(w) \tag{2.2}$$

is satisfied.

For every $v \in \mathbb{V}$, we set $\|v\| = L(v)$, and we call it the *Minkowski norm* of v. Notice that we do not assume the reversibility condition $\|v\| = \|-v\|$, thus $\| \bullet \|$ is not a norm in the usual sense. Therefore the indicatrix I defined by

$$I = \{v \in \mathbb{V} | \|v\| = 1\} \tag{2.3}$$

is a hypersurface in $\mathbb{V}$ which is not symmetric around the origin in general. The pair $(\mathbb{V}, L)$ or $(\mathbb{V}, \| \bullet \|)$ is called a *Minkowski space.*

Definition 2.2. Let $(\mathbb{V}_1, L_1)$ and $(\mathbb{V}_2, L_2)$ be two Minkowski spaces. A map $P : \mathbb{V}_1 \longrightarrow \mathbb{V}_2$ is called a *norm-preserving map* if it satisfies $L_1(v) = L_2(P(v))$, that is,

$$\|v\|_1 = \|P(v)\|_2 \tag{2.4}$$

for every $v \in \mathbb{V}_1$. We also call a map $P : \mathbb{V}_1 \longrightarrow \mathbb{V}_2$ an *isometry* if P satisfies

$$\|v - w\|_1 = \|P(v) - P(w)\|_2 \tag{2.5}$$

for every $v, w \in \mathbb{V}_1$. If there exists an isometry $P : \mathbb{V}_1 \longrightarrow \mathbb{V}_2$, we say that $(\mathbb{V}_1, L_1)$ is *isometric* or *congruent* to $(\mathbb{V}_2, L_2)$.

If an isometry $P : \mathbb{V}_1 \longrightarrow \mathbb{V}_2$ satisfies $P(0) = 0$, then by substituting $w = 0$ in (2.5), we obtain (2.4). Therefore any isometry is a norm-preserving map.

Any norm-preserving map $P : \mathbb{V}_1 \longrightarrow \mathbb{V}_2$ satisfies $P(0) = 0$, but not an isometry in general. If a norm-preserving map $P : \mathbb{V}_1 \longrightarrow \mathbb{V}_2$ is linear, then it is trivial that P is an isometry.

Let $\mathbb{V}$ be a vector space with a Minkowski norm $\| \bullet \|$. We set

$$\|v\|_0 = \frac{1}{2}(\|v\| + \| - v\|) = \frac{1}{2}[L(v) + L(-v)]$$

for every $v \in \mathbb{V}$. Then $\| \bullet \|_0$ is a norm in the usual sense, that is, the following condition are satisfied.

(1) For every $v \in \mathbb{V}, \|v\|_0 \geqslant 0$ is satisfied, and the equality holds if and only if $v = 0$.

(2) For every $v \in \mathbb{V}$ and every $\lambda \in \mathbb{R}$

$$\|\lambda v\|_0 =| \lambda | \, \|v\|_0 \tag{2.6}$$

is satisfied.

(3) For any $v, w \in \mathbb{V}$, the triangle inequality

$$\|v - w\|_0 \leq \|v\|_0 + \| - w\|_0 \tag{2.7}$$

is satisfied.

Assume that $P : \mathbb{V}_1 \longrightarrow \mathbb{V}_2$ is an isometry. Then using the symmetrized norm we have

$$\begin{aligned}\|P(v) - P(w)\|_0 &= \frac{1}{2}(\|P(v) - P(w)\|_2 + \| - P(v) + P(w)\|_2)\\ &= \frac{1}{2}(\|v - w\|_1 + \| - v + w\|_1)\\ &= \|v - w\|_0\end{aligned}$$

Therefore, if P is an isometry between Minkowski spaces, then P is also an isometry with respect to the usual norm $\| \bullet \|_0$. Consequently, the Mazur-Ulam theorem, we have following

Theorem 2.1. (Mazur-Ulam). *If $\mathbb{V}_1$ and $\mathbb{V}_2$ are Minkowski spaces and if P is an isometry from $\mathbb{V}_1$ onto $\mathbb{V}_2$ with $P(0) = 0$, then P is linear.*

2.2 Finsler metrics

Let M be a smooth connected manifold of dim $M = n$, and $\pi : TM \longrightarrow M$ its tangent bundle over M. We use the chart $(\pi^{-1}(U), (x^i, y^i)_{(1\leq i\leq n)}$ on TM induced by a chart $(U, x^i)_{(1\leq i\leq n)}$ on M, where $y_1, \cdots, y^n$ are the fibre coordinates in each $T_xM, x \in U$.

Definition 2.3. A function $L : TM \to \mathbb{R}$ is called a *Finsler metric* or length function on M if L satisfies the following conditions.

(1) L satisfies $L(x, y) \geq 0$ for every $y \in T_xM$, and the equality holds if and only if $y = 0$.

(2) For every $y \in T_xM$ and $\lambda \in \mathbb{R}^+$, the homogeneity condition $L(x, \lambda y) = \lambda L(x, y)$ holds at each point $x \in M$.

(3) L is continuous on TM, and L is smooth on the slit tangent bundle $TM \setminus \{0_M\}$.

(4) For any $y_1, y_2 \in T_xM$, the triangle inequality

$$L(x, y_1 + y_2) \leq L(x, y_1) + L(x, y_2) \tag{2.8}$$

holds at each point $x \in M$.

The pair (M, L) is called a *Finsler space*.

For every tangent vector $y \in T_xM$, we set

$$\|y\|_L \stackrel{\text{def}}{=} L(x, y). \tag{2.9}$$

Then each tangent space T_xM at $x \in M$ is regarded as a Minkowski space with the Minkowski norm $\| \bullet \|_L = L(x, \bullet)$, where x is fixed.

Definition 2.4. The hypersurface $I_x = \{y \in T_xM | L(x,y) = 1\}$ in each tangent space T_xM is called the indicatrix at $x \in M$ of (M, L).

Definition 2.5. A Finsler metric L on M is said to be strongly-convex if the $n \times n$-matrix (G_{ij}) defined by the Hessian

$$G_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j} \tag{2.10}$$

is positive-definite at each point of $\pi^{-1}(U)$

If L is strongly-convex, then the Hessian (G_{ij}) defined by (2.10) induces an inner product $G_{(x,y)}$ on the fiber $V_{(x,y)} = T_y(T_xM)$ of the vertical sub-bundle $V \subset TTM$ by

$$G_{(x,y)}\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = G_{ij}(x,y). \tag{2.11}$$

Let $c : I = [0,1] \to M$ be a smoot curve with the starting point $p = c(0)$ and the terminal point $q = c(1)$. A smooth curve $c = c(t)$ is said to be *regular* if $\dot{c}(t) := dc/dt \neq 0$ for every $t \in I$. The *length* $l(c)$ of a curve $c = c(t)$ is defined by

$$l(c) = \int_0^1 \|\dot{c}(t)\| dt = \int_0^1 L(c(t), \dot{c}(t)) dt \tag{2.12}$$

Since L satisfies the homogeneity condition, this definition is well-defined, that is, the length of a curve c is invariant by any change of parameter t which preserves the orientation of c.

For the set $\Gamma(p,q)$ of all regular oriented curves from the starting point p to the terminal point q, we define a functional $\mathcal{F}_L : \Gamma(p,q) \to \mathbb{R}$ by

$$\mathcal{F}_L(c) = \int_0^1 \|\dot{c}(t)\|_L dt = \int_0^1 L(c(t), \dot{c}(t)) dt \tag{2.13}$$

For an ordered pair $(p,q) \in M \times M$, we define a function

$$d_L(p,q) = \inf_{c \in \Gamma(p,q)} \mathcal{F}_L(c).$$

The function d_L satisfies the following conditions:

(1) $d_L(p,q) \geq 0$,

(2) $d_L(p,q) = 0$ if and only if $p = q$,

(3) $d_L(p,q) \leq d_L(p,r) + d_L(r,q)$.

Since the reversibility condition $L(x,y) = L(x,-y)$ is not assumed, the reversibility condition $d_L(p,q) = d_L(q,p)$ is not satisfied in general. Thus d_L is a *pseudo-distance* on M.

2.3 Geodesics in Finsler manifolds

Let $c(t) = (x^1(t), \cdots, x^n(t))$ be an oriented regular curve on a smooth manifold of dim $M = n$. If a strongly-convex Finsler metric L is given on M, the length of c is defined by (2.12). A curve in (M, L) is called a *geodesic* if it is locally a distance-minimizing curve. For any curve $c = c(t) \in \mathit{\Gamma}(p,q)$, with the starting point $p = c(0)$ and the terminal point $q = c(1)$, and for a sufficiently small $\varepsilon(-\varepsilon < s < \varepsilon)$, we take a variation $\mathit{\Gamma}_c : c_s(t) = c(t) + sX$ of c, where $X = X(t)$ is any smooth vector field defined along the curve c satisfying $X(p) = X(q) = 0$. Then, since $c_0(t) = c(t)$, we have

$$\|\dot{c}_s(t)\|_L - \|\dot{c}(t)\|_L = s\left(\sum \frac{\partial L}{\partial x^i} X^i + \sum \frac{\partial L}{\partial y^i} \dot{X}^i\right) + \frac{s^2}{2}(\cdots) + \cdots .$$

Therefore we obtain

$$\left.\frac{d}{ds}\right|_{s=0} \mathcal{F}_L(c_s) = \int_0^1 \left(\sum \frac{\partial L}{\partial x^i} X^i + \sum \frac{\partial L}{\partial y^i} \dot{X}^i\right) dt.$$

On the other hand

$$\frac{d}{dt}\left(\sum \frac{\partial L}{\partial y^i} X^i\right) = \sum \frac{\partial L}{\partial y^i} \dot{X}^i + \frac{d}{dt}\frac{\partial L}{\partial y^i} X^i$$

implies

$$\begin{aligned}
\left.\frac{d}{ds}\right|_{s=0} \mathcal{F}_L(c_s) &= \int_0^1 \left[\sum \frac{\partial L}{\partial x^i} X^i + \frac{d}{dt}\left(\sum \frac{\partial L}{\partial y^i} X^i\right) - \frac{d}{dt}\left(\sum \frac{\partial L}{\partial y^i}\right) X^i\right] dt \\
&= \left[\sum \frac{\partial L}{\partial y^i} X^i\right]_0^1 + \int_0^1 \sum \left[\frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial y^i}\right)\right] X^i dt \\
&= \int_0^1 \sum \left[\frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial y^i}\right)\right] X^i dt = 0.
\end{aligned}$$

Consequently a curve $c = c(t) \in \mathit{\Gamma}(p,q)$ is a critical point of the functional $\mathcal{F}_L$ if and only if

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial y^i}\right) = 0 \tag{2.14}$$

is satisfied along c. This differential equation is called the *Euler-Lagrange equation* of the functional $\mathcal{F}_L$.

Definition 2.6. A smooth curve $c = c(t)$ is called a *geodesic* if (2.14) is satisfied along c.

Suppose that $\gamma = \gamma(t) \in \Gamma(p, q)$ is minimizing the functional $\mathcal{F}_L$, that is, γ satisfies $\mathcal{F}_L(c) \geq \mathcal{F}_L(\gamma)$ for all $c \in \Gamma(p, q)$. Therefore we have

$$d_L(p, q) = \mathcal{F}_L(\gamma). \tag{2.15}$$

Then γ is a critical point of the variation $\mathcal{F}_L(c_s)$, and the Euler-Lagrange equation (2.14) is satisfied along γ. Consequently γ is a geodesic in (M, L).

Proposition 2.1. *If $\gamma \in \Gamma(p, q)$ is an $\mathcal{F}_L$-minimizing curve, then γ is a geodesic.*

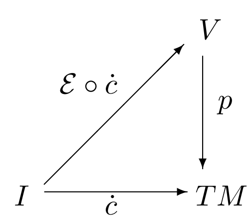
In the sequel, we set $F = L^2 = \sum G_{ij}(x, y)y^i y^j$ and we treat the energy functional $\mathcal{F}_G$ depends on the parametrization unlike the functional $\mathcal{F}_L$,

$$\mathcal{F}_G(c) = \frac{1}{2}\int_a^b \widetilde{c}_{\dot{c}}^{*} G(\mathcal{E}, \mathcal{E})dt, \qquad (0 \leq a \leq b \leq 1)$$

where $\widetilde{c}_{\dot{c}}^{*} G(\mathcal{E}, \mathcal{E})$ is given by

$$\widetilde{c}_{\dot{c}}^{*} G(\mathcal{E}, \mathcal{E}) = G_{(c(t), \dot{c}(t))}(\dot{c}(t)^V, \dot{c}(t)^V) = L(c(t), \dot{c}(t))^2 \tag{2.16}$$

and $\dot{c}(t)^V = (\mathcal{E} \circ \dot{c})(t)$ is the vertical lift of $\dot{c}$ along the canonical lift of $\widetilde{c}_{\dot{c}} = (c(t), \dot{c}(t))$.



Since $\widetilde{c}_{\dot{c}}^{*} G(\mathcal{E}, \mathcal{E}) = L(c(t), \dot{c}(t))^2$, the Cauchy-Schwarz inequality

$$\left(\int_a^b L(c(t), \dot{c}(t))dt\right)^2 \leq \int_a^b L(c(t), \dot{c}(t))^2 dt \cdot \int_a^b 1^2 dt$$

yields

$$\mathcal{F}_L(c)^2 \leq 2(b-a)\mathcal{F}_G(c), \tag{2.17}$$

where the equality holds if and only if $\|\dot{c}(t)\|_L = L(c(t), \dot{c}(t))$ is constant, that is, the parameter t is normal, since

$$L(c(t), \dot{c}(t)) = L\left(c(s), \frac{dc}{ds}\frac{ds}{dt}\right) = \frac{ds}{dt}.$$

Suppose that a curve $\gamma = \gamma(t)$ with normal parameter t is $\mathcal{F}_G$-minimizing, then we have

$$\begin{aligned} \mathcal{F}_L(c) &= \{2(b-a)\mathcal{F}_G(c)\}^{1/2} \\ &\geq \{2(b-a)\mathcal{F}_G(\gamma)\}^{1/2} \\ &= \mathcal{F}_L(\gamma) \end{aligned}$$

for all $c \in \Gamma(p, q)$.

Proposition 2.2. *If a regular curve γ in M is $\mathcal{F}_G$-minimizing, then γ is a geodesic in (M, L).*

Therefore it is enough to investigate curves minimizing the energy functional $\mathcal{F}_G$. The Euler-Lagrange equation of the functional $\mathcal{F}_G$ is given by

$$\frac{\partial F}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial F}{\partial y^i}\right) = 2L\left[\frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial y^i}\right)\right] - 2\frac{\partial L}{\partial y^i}\frac{\partial L}{dt}. \tag{2.18}$$

If we change the parameter t to the arc-length s, that is, if we assume

$$L\left(x, \frac{dx}{ds}\right) \equiv 1,$$

then equation (2.14) is written as

$$\frac{\partial F}{\partial x^i} - \frac{d}{ds}\left(\frac{\partial F}{\partial y^i}\right) = 0. \tag{2.19}$$

This equation is computed as follows:

$$\begin{aligned} \frac{d}{ds}\left(\frac{\partial F}{\partial y^i}\right) - \frac{\partial F}{\partial x^i} &= \frac{d}{ds}\left(\sum G_{ij}y^j\right) - \frac{1}{2}\sum \frac{\partial G_{jk}}{\partial x^i}y^j y^k \\ &= \sum \frac{\partial G_{ij}}{\partial x^k}y^j y^k + \sum G_{ij}\frac{d^2x^j}{ds^2} - \frac{1}{2}\sum \frac{\partial G_{jk}}{\partial x^i}y^j y^k \\ &= \sum G_{ij}\frac{d^2x^j}{ds^2} + \frac{1}{2}\left(\frac{\partial G_{ij}}{\partial x^k} + \frac{\partial G_{ik}}{\partial x^j} - \frac{\partial G_{jk}}{\partial x^i}\right)\frac{dx^j}{ds}\frac{dx^k}{ds}. \end{aligned}$$

Therefore we have

Proposition 2.3. *The differential equation of geodesics is given by*

$$\frac{d^2x^i}{ds^2} + \sum \left\{ {i \atop j\,k} \right\} (\gamma(s), \gamma'(s)) \frac{dx^j}{ds} \frac{dx^k}{ds} = 0, \tag{2.20}$$

where s is the arc-length with respect to the Finsler metric L and

$$\left\{ {i \atop j\,k} \right\}(x, y) = \frac{1}{2} \sum G^{ir} \left(\frac{\partial G_{rj}}{\partial x^k} + \frac{\partial G_{rk}}{\partial x^j} - \frac{\partial G_{jk}}{\partial x^r} \right), \tag{2.21}$$

are the Christoffel symbols of the metric tensor $G = (G_{ij})$ and (G^{ij}) is the inverse matrix of (G_{ij}).

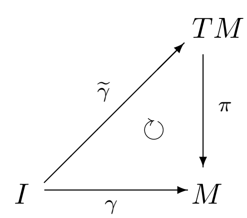
If we define a local function G^i by

$$G^i(x, y) = \frac{1}{2} \sum \left\{ {i \atop j\,k} \right\} (x, y) y^j y^k, \tag{2.22}$$

then (2.20) is written as follows

$$\frac{d^2x^i}{ds^2} + 2G^i \left(x, \frac{dx}{ds} \right) = 0. \tag{2.23}$$

Let $\gamma = \gamma(s)$ be a geodesic in a Finsler manifold (M, L) and $\widetilde{\gamma}(s) = (\gamma(s), \gamma'(s))$ the natural lift of γ to the total space TM, where the parameter s is the arc-length with respect to the metric L.



The velocity vector field of the natural lift of a geodesic $\gamma(s)$ is given by

$$\frac{d\widetilde{\gamma}}{ds} = \sum \left(\frac{dx^i}{ds} \frac{\partial}{\partial x^i} + \frac{d^2x^i}{ds^2} \frac{\partial}{\partial y^i} \right) = \sum \frac{dx^j}{ds} \frac{\partial}{\partial x^j} - \sum 2G^i \left(x, \frac{dx}{ds} \right) \frac{\partial}{\partial y^i}.$$

The functions G^i defined by (2.22) are homogeneous of degree two, that is, $2G^i = \sum \frac{\partial G^i}{\partial y^j} y^j$ holds, and thus

$$2G^i \left(x, \frac{dx}{ds} \right) = \sum \frac{\partial G^i}{\partial y^j} \left(x, \frac{dx}{ds} \right) \frac{dx^j}{ds}.$$

Consequently we have

$$\frac{d\widetilde{\gamma}}{ds} = \sum \frac{dx^j}{ds}\left[\frac{\partial}{\partial x^j} - \sum \frac{\partial G^i}{\partial y^j}\left(x, \frac{dx}{ds}\right)\frac{\partial}{\partial y^i}\right].$$

Since $d\pi\left(\frac{d\widetilde{\gamma}}{ds}\right) = \frac{d\gamma}{ds} \neq 0$, this shows that

$$\frac{d\widetilde{\gamma}}{ds} \in TTM/V \tag{2.24}$$

at each point in $\widetilde{\gamma}(s) \in TM$. Therefore we determine the horizontal sub-bundle $H \subset TTM$ so that the velocity vector field of $\widetilde{\gamma}(s)$ always lies in the horizontal space at each point $\widetilde{\gamma}(s)$:

$$\frac{d\widetilde{\gamma}}{ds} \in H_{\widetilde{\gamma}(s)}.$$

For the function $G^i(x,y)$ given by (2.22), we define

$$N^i_j(x,y) \stackrel{\text{def}}{=} \frac{\partial G^i}{\partial y^j} \tag{2.25}$$

and $\theta \in A^1(V)$ by

$$\theta = \sum \frac{\partial}{\partial y^i} \otimes \theta^i \stackrel{\text{def}}{=} \sum \frac{\partial}{\partial y^i} \otimes \left(dy^i + \sum N^i_j(x,y)dx^j\right). \tag{2.26}$$

Then θ defines a non-linear connection on the tangent bundle TM of (M,L).

Definition 2.7. The non-linear connection θ defined by (2.26) is called the *Berwald non-linear connection* of (M,L).

2.4 Examples of Finsler manifolds

Let M be a smooth manifold with a Riemannian metric $g = \sum g_{ij}(x)dx^i \otimes dx^j$.

2.4.1 Riemannian metrics

For every $y \in T_xM$, if we define a function $L : TM \longrightarrow \mathbb{R}$ by

$$L(x,y) := \sqrt{\sum g_{ij}(x)y^iy^j}. \tag{2.27}$$

The norm $\|X\|$ of any vector filed X in M is measured by

$$\|X\| = \sqrt{g(X,X)}.$$

Then L is a Finsler metric on M, and therefore any Riemannian manifold belongs to the class of Finsler spaces.

Each indicatrix $I_x = \{y \in T_xM | \sqrt{g(y,y)} = 1\}$ at $x \in M$ is considered as the unit sphere with the center $y = 0$ in T_xM, since around each $x \in M$ we may choose a local orthonormal frame field. As stated in the previous chapter, the parallel transport P_γ along any curve $\gamma : [a,b] \to M$ with respect to the Levi-Civita connection ∇^g is a linear isometric map from the Euclidean space $(T_{\gamma(a)}M, g_{\gamma(a)})$ to the one $(T_{\gamma(b)}M, g_{\gamma(b)})$. Thus the unit sphere $I_{\gamma(a)}$ is also linear isometric to $I_{\gamma(b)}$. A Riemannian manifold (M,g) is a space *modeled on a unique inner product space.*

2.4.2 Randers metrics

A simplest modification of a Riemannian metric $g = \sum g_{ij}(x)dx^i \otimes dx^j$ was introduced by Randers from the physical view point. Let $\beta = \sum \beta_i(x)dx^i$ be a differential one-form on M whose norm $\|\beta\|_g$ with respect to g satisfies $\|\beta\|_g < 1$. We define a function $L : TM \longrightarrow \mathbb{R}$ by

$$L(x,y) = \sqrt{\sum g_{ij}(x)y^iy^j} + \sum \beta_i(x)y^i. \tag{2.28}$$

Such a non-Riemannian Finsler metric L on M is called a *Randers metric* (cf. [Ma1]). The norm $\|X\|$ of any vector filed X in M is measured by

$$\|X\| = \sqrt{g(X,X)} + \beta(X).$$

A Randers metric is an asymmetrical modification of g because of $L(x,y) \neq L(x,-y)$. Such a metric is characterized as a metric such that each indicatrix I_x is a quadratic hypersurface in each tangent space T_xM whose center is not the origin $y = 0$ of T_xM ([Ha-Ic]).

Chapter 3

Berwarld connections

3.1 Inner product on V

Let L be a strongly-convex Finsler metric on a smooth manifold M. Since the smoothness of L at the zero-section is not assumed, every quantity obtained from L is not smooth on the whole total space TM. In this section we shall show that a natural Ehresmann connection θ can be introduced on TM from the given Finsler metric L.

Let $\widetilde{TM}$ be the pullback of TM by $\pi : TM \longrightarrow M$

$$\begin{array}{ccc} \widetilde{TM} & \xrightarrow{\ \widetilde{\pi}\ } & TM \\ \big\downarrow & \circlearrowright & \big\downarrow \pi \\ TM & \xrightarrow[\ \pi\ ]{} & M \end{array}$$

Since the induced bundle $\widetilde{TM}$ is isomorphic to both the horizontal sub-bundle H and the vertical sub-bundle V of $T^2M := TTM$, we shall consider both an Ehresmann connection θ and the derivative $d\pi$ as projections from T^2M onto V, and we use the notations introduced in the first chapter.

Let $(x^1, \cdots, x^n)$ be a local coordinate in an open subset U in M. With respect to the natural local frame field $\dfrac{\partial}{\partial x} = \left\{ \dfrac{\partial}{\partial x^1}, \cdots, \dfrac{\partial}{\partial x^n} \right\}$ over U, any tangent vector $y \in T_xM$ is written as

$$y = \sum y^i \left(\frac{\partial}{\partial x^i} \right)_x .$$

Then $(x^1, \cdots, x^n, y^1, \cdots, y^n)$ induces a local coordinate on $\pi^{-1}(U) \subset TM$. Then the projections

θ and $d\pi$ are expressed as

$$\theta = \sum \frac{\partial}{\partial y^i} \otimes \theta^i \quad \text{and} \quad d\pi = \sum \frac{\partial}{\partial y^i} \otimes dx^i$$

respectively.

If a strongly-convex Finsler metric L is given on M, there exists an inner product G on the vertical sub-bundle V defined by (2.11). By the homogeneity condition of L the following identity holds:

$$L(x,y)^2 = \sum G_{ij}(x,y) y^i y^j = G(\mathcal{E}, \mathcal{E}), \tag{3.1}$$

for the tautological section $\mathcal{E}$ of V. Since we do not assume the smoothness of L at the zero-section $y = 0$ in TM, the Hessian G_{ij} is not smooth at $y = 0$.

Since each fibre $V_{(x,y)}$ of V over $(x,y) \in TM$ is identified with the tangent space $T_y(T_xM)$, the inner product $G_{(x,y)}$ on $V_{(x,y)}$ may be considered as a Riemannian metric on the tangent space T_xM. Thus we call the tangent spaces with such a Riemannian metric G the *tangential Riemannian spaces* of (M, L)

3.2 Berwald connection

Let θ be the Berwald non-linear connection of a Finsler manifold (M, L), namely, θ is defined by (2.26) for the coefficients $N^i_j(x,y)$ given by (2.25).

Definition 3.1. The canonical connection $D : \Gamma(V) \to A^1(V)$ in V associated with the Berwald non-linear connection θ is called the *Berwald connection* on (M, L).

In the sequel, on a Finsler manifold (M, L), we always take the Berwald non-linear connection θ, especially unless otherwise stated. The coefficients of a connection form $\omega^i_j = \sum \Gamma^i_{jk}(x,y)dx^k$ are given by

$$\Gamma^i_{jk}(x,y) = \frac{\partial N^i_j}{\partial y^k} = \frac{\partial^2 G^i}{\partial y^j \partial y^k},$$

and the equation (2.20) of geodesics is written as follows:

$$\frac{d^2x^i}{ds^2} + \sum \Gamma^i_{jk}\left(x, \frac{dx}{ds}\right) \frac{dx^j}{ds}\frac{dx^k}{ds} = 0. \tag{3.2}$$

Since the connection coefficients Γ^i_{jk} satisfy the symmetry condition $\Gamma^i_{jk} = \Gamma^i_{kj}$, the projection $d\pi$ is parallel with respect to D :

$$Dd\pi = 0. \tag{3.3}$$

Proposition 3.1. *The Berwald connection D is symmetric.*

Since the Berwald connection D is the canonical connection associated with θ, we have

$$D_{X^H}\mathcal{E} = 0$$

for any vector field X in M. The homogeneity of N^i_j implies $N^i_k = \sum y^j \Gamma^i_{jk}$ and thus

$$\begin{aligned} X^H(L^2) &= X^H(G(\mathcal{E},\mathcal{E})) \\ &= X^H(\sum G_{ij}\,(x,y) y^i y^j) \\ &= \sum X^k \left[\left(\frac{\partial G_{ij}}{\partial x^k} - \sum N^l_k \frac{\partial G_{ij}}{\partial y^l} \right) y^i y^j - 2 \sum G_{ij} y^i N^j_k \right] \\ &= \sum X^k \left(\frac{\partial G_{ij}}{\partial x^k} - 2 \sum G_{lj} \left\{ {l \atop i\,k} \right\} \right) y^i y^j \\ &= 0. \end{aligned}$$

Consequently we obtain

$$\mathcal{L}_{X^H} L = X^H(L) \equiv 0. \tag{3.4}$$

Proposition 3.2. *The Berwald connection D on a Finsler manifold (M, L) is almost L-metrical.*

Let $c : I \to M$ be a smooth curve in M. It may be assumed without loss of generality that c is a regular curve. Suppose that a vector field X on M is parallel along c with respect to the Berwald non-linear connection θ. Then, X is parallel along c if and only if

$$c^*(X^*\theta)\left(\frac{d}{dt}\right) = 0.$$

This equation is written as

$$\frac{dX^i}{dt} + \sum N^i_j(c(t), X(t)) \frac{dx^j}{dt} = 0. \tag{3.5}$$

Since the norm $\|X(t)\|_L$ of $X(t)$ is given by $\|X(t)\|_L = L(c(t), X(t))$, if X is parallel along c, then (3.4) implies

$$\frac{d}{dt}\|X(t)\|_L = \frac{d}{dt} L(c(t), X(t)) = 0$$

Therefore we have

Proposition 3.3. *Let c be any smooth curve with initial point p in a Finsler manifold (M, L). Then the parallel translation P_c along c is a norm-preserving map, i.e., for any $Y \in T_pM$ we have*

$$\|Y\|_L = \|P_c(Y)\|_L. \tag{3.6}$$

In particular P_c satisfies $P_c(0) = 0$.

The indicatrix I_x is a compact hypersurface in T_xM given by the set of tangent vectors of unit norm. Since P_c preserves the norm, we have $\|P_c(y)\|_L = \|y\|_L = 1$ for any $y \in I_x$. Therefore we obtain

Proposition 3.4. *In a Finsler manifold (M, L), the parallel translation P_c along any smooth curve c preserves the indicatrix, i.e.,*

$$I_{\varphi_t(x)} = \varphi_t^H(I_x).$$

3.3 Curvature and torsion of Berwald connection

The curvature R_D of the Berwald connection D on a Finsler manifold (M, L) is defined by $R_D = D^2$. Since D is a canonical connection associated with θ, the curvature R_D is decomposed into the sum $R_D = R_D^{HH} + R_D^{HV}$, where R_D^{HH} and R_D^{HV} are defined by

$$R_D^{HH}(X, Y)Z^V := R_D(X^H, Y^H)Z^V = D_{X^H}D_{Y^H}Z^V - D_{Y^H}D_{X^H}Z^V - D_{[X^H,Y^H]}Z^V$$

and

$$R_D^{HV}(X, Y)Z^V := R_D(X^H, Y^V)Z^V = D_{X^H}D_{Y^V}Z^V - D_{Y^V}D_{X^H}Z^V - D_{[X^H,Y^V]}Z^V$$

for all vector fields X, Y and Z in M. The components R^i_{jkl} of Ω_D^{HH} are given by

$$R^i_{jkl} = \left(\frac{\partial}{\partial x^k}\right)^H \Gamma^i_{jl} - \left(\frac{\partial}{\partial x^l}\right)^H \Gamma^i_{jk} + \sum \Gamma^i_{mk}\Gamma^m_{jl} - \sum \Gamma^i_{ml}\Gamma^m_{jk}, \tag{3.7}$$

and, furthermore the components P^i_{jkl} of Ω_D^{HV} are given by

$$P^i_{jkl} = -\frac{\partial \Gamma^i_{jk}}{\partial y^l} = -\frac{\partial^2 N^i_j}{\partial y^k \partial y^l} = -\frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}. \tag{3.8}$$

Therefore the curvature form $\Omega_D = (\Omega^i_j)$ of D is given by

$$\Omega^i_j = \sum_{k<l} R^i_{jkl}(x,y)dx^k \wedge dx^l + \sum P^i_{jkl}(x,y)dx^k \wedge \theta^l. \tag{3.9}$$

From (3.7) and (3.8), these coefficients satisfy $R^i_{jkl} \equiv R^i_{jlk}$ and $P^i_{jkl} = P^i_{jlk}$, or equivalently

$$R^{HH}(X,Y) = -R^{HH}(Y,X), \qquad R^{HV}(X,Y) = R^{HV}(Y,X) \tag{3.10}$$

for all vector fields X, Y in M.

First we shall consider the case where the curvature R_D vanishes identically.

Definition 3.2. A Finsler manifold (M, L) is said to be *locally Minkowski* or *flat* if there exists an open covering of M with respect to which the metric L is independent of the base point $x \in M$.

The aim of this section is to characterize locally Minkowski spaces in terms of curvature R_D of the Berwald connection D. Before proving the main theorem, we shall show the transformation laws of curvature forms with respect to a coordinate change in the base space M. Let U and $\overline{U}$ be two coordinate neighborhoods in M with local coordinate $(x^1, \cdots, x^n)$ and $(\overline{x}^1, \cdots, \overline{x}^n)$ respectively such that $U \cap \overline{U} \neq 0$. The relations between the respective fibre coordinates $(y^1, \cdots, y^n)$ and $(\overline{y}^1, \cdots, \overline{y}^n)$ relative to $(x^1, \cdots, x^n)$ and $(\overline{x}^1, \cdots, \overline{x}^n)$ are given by $\overline{y}^i = \sum \frac{\partial \overline{x}^i}{\partial x^l} y^l$. Considering $y = {}^t(y^1, \cdots, y^n)$ and $\overline{y} = {}^t(\overline{y}^1, \cdots, \overline{y}^n)$ as column vectors, we write this relations as $\overline{y} = Ay$, where we set $A = \left(\frac{\partial \overline{x}^i}{\partial x^l}\right)$. Then, using matrix notations, the natural local frame fields $e = \left\{\frac{\partial}{\partial y^1}, \cdots, \frac{\partial}{\partial y^n}\right\}$ and $\overline{e} = \left\{\frac{\partial}{\partial \overline{y}^1}, \cdots, \frac{\partial}{\partial \overline{y}^n}\right\}$ of V are related to the following equation

$$e_U = \overline{e}A. \tag{3.11}$$

Therefore the respective connection forms ω and $\overline{\omega}$ of D relative to e and $\overline{e}$ are related as

$$\omega = A^{-1}(dA + \overline{\omega}A). \tag{3.12}$$

We express the curvature R_D as

$$R_D = e \otimes \Omega = \overline{e} \otimes \overline{\Omega}$$

for the respective curvature forms Ω and $\overline{\Omega}$ of D relative to c and $\bar{c}$. Then (3.12) implies

$$\Omega = A^{-1}\overline{\Omega}A.$$

Suppose that (M, L) is locally Minkowski. Then the metric L is independent of local coordinate $\overline{x} = (\overline{x}^1, \cdots, \overline{x}^n)$ in M. Then the components $\overline{G}_{ij}$ of the metric tensor G in V with respect to this local coordinate $(\overline{x}, \overline{y})$ are also independent of $\overline{x}$. This fact implies $\overline{\left\{ {i \atop lm} \right\}} = 0$. Therefore we have

$$\overline{\Gamma}^i_{jk} = \frac{1}{2}\frac{\partial^2}{\partial \overline{y}^j \overline{y}^k}\left[\sum \overline{\left\{ {i \atop lm} \right\}} \overline{y}^l \overline{y}^m\right] = 0,$$

that is, $\overline{\omega} = 0$ in $\overline{U}$. This shows that $R_D \equiv 0$.

Conversely we suppose $R_D \equiv 0$. This assumption is the integrability condition for the system of differential equations

$$dB = B\omega, \tag{3.13}$$

where $B = (B^i_j)$ is a certain local function with values in $GL(n, \mathbb{R})$. In fact,

$$0 = d(dB) = d(B\omega) = dB \wedge \omega + Bd\omega = B(d\omega + \omega \wedge \omega) = B\Omega_D.$$

Furthermore $\omega \wedge dx = 0$ is satisfied, since D is symmetric. Then, from (3.7) we have $dB \wedge dx = 0$, namely

$$\frac{\partial B^i_j}{\partial x^k} = \frac{\partial B^i_k}{\partial x^j}$$

in local coordinates. Therefore there exist some local functions $f^i(x^1, \cdots, x^n)$ such that $B^i_j = \dfrac{\partial f^i}{\partial x^j}$. Then, if we take a change of local coordinate as $\overline{x}^i = f^i(x^1, \cdots, x^n)$, the connection form $\overline{\omega}$ of D with respect to $(\overline{x}, \overline{y})$ vanishes. Then, from (3.4), we have

$$0 = \left(\frac{\partial}{\partial \overline{x}^i}\right)^H L = \frac{\partial L}{\partial \overline{x}^i} - \sum \overline{y}^m \overline{\Gamma}^l_{mi}\frac{\partial L}{\partial \overline{y}^l} = \frac{\partial L}{\partial \overline{x}^i}.$$

Therefore L is independent of the local coordinate $(\overline{x}^1, \cdots, \overline{x}^n)$ in M. Consequently we obtain

Theorem 3.1. *A Finsler manifold (M, L) is locally Minkowski if and only if its Berwald connection D is flat, that is, $R^{HH}_D = R^{HV}_D \equiv 0$.*

Because of (3.3), the projection $d\pi$ from T^2M onto V is parallel with respect to D. The torsion T_D of the Berwald connection D is defined by :

$$T_D = D\theta. \tag{3.14}$$

Since D is the canonical connection, the torsion T_D is of the form $T_D = T_D^{HH}$, where T_D^{HH} is given by

$$T_D^{HH} = \sum \frac{\partial}{\partial y^i} \otimes \left(\sum_{k<l} R^i_{kl}(x,y) dx^k \wedge dx^l \right), \tag{3.15}$$

where the coefficients R^i_{kl} are given by

$$R^i_{kl} = \left(\frac{\partial}{\partial x^k} \right)^H N^i_l - \left(\frac{\partial}{\partial x^l} \right)^H N^i_k. \tag{3.16}$$

Since $T_D^{HV} \equiv 0$, the Ricci identity $D^2\mathcal{E} = R_D\mathcal{E}$ implies

Proposition 3.5. *The curvature R_D of D satisfies the following identities*

$$R_D^{HH}\mathcal{E} = T_D^{HH} \tag{3.17}$$

and

$$R_D^{HV}\mathcal{E} = 0 \tag{3.18}$$

for the tautological section $\mathcal{E}$.

3.4 Berwald spaces

As stated in the previous chapter, if a strongly convex Finsler metric L is given on M, then each tangent space T_xM has two metrical structures. One is the normed space with the Minkowski norm $\|X\|_L = L(x, X)$ for any $X \in T_xM$, and another one is the inner-product space with the Riemannian metric G on T_xM defined by (2.11). In this section we shall consider the case where the parallel translation P_c with respect to the Berwald non-linear connection θ preserves these structures.

First we shall consider the case where the parallel translation P_c is an isometry between the tangential Minkowski spaces. By Definition 2.2 this condition is written as

$$\|X - Y\|_L = \|P_c(X) - P_c(Y)\|_L$$

for any $X, Y \in T_pM$. Then, from Theorem 2.1 in the previous chapter, P_c is a linear isomorphism. The converse is also true. Therefore we have

Proposition 3.6. *The parallel translation P_c along any curve c is an isometry between the tangential Minkowski spaces if and only if P_c is a linear isomorphism.*

Definition 3.3. A Finsler manifold (M, L) is called a *Berwald space* if the parallel translation P_c along any curve c is a linear isomorphism.

Therefore, from Proposition 3.6 we have

Proposition 3.7. *A Finsler manifold (M, L) is a Berwald space if and only if its Berwald connection D is induced from a linear connection θ in TM.*

A necessary and sufficient condition for the parallel translation P_c to be a linear isomorphism is that the coefficients N^i_j of θ defined in (2.25) are linear in fibre coordinate $(y^1, \cdots, y^n)$, that is, the Berwald non-linear connection θ is reduced to a linear connection on TM. The Berwald connection D is induced from a linear connection ∇^M on TM. Thus, using the connection coefficients $\gamma^i_{jk}(x)$ of ∇^M, the coefficients N^i_j of θ is written as

$$N^i_j(x, y) = \sum \gamma^i_{jk}(x) y^k, \tag{3.19}$$

which implies $\varGamma^i_{jk} = \gamma^i_{jk}(x)$. From (3.8) we obtain

Proposition 3.8. *A Finsler manifold (M, L) is a Berwald space if and only if its Berwald connection D satisfies $R^{HV} \equiv 0$.*

If (M, L) is a Berwald space, the connection coefficients $\varGamma^i_{jk}$ of D are independent of the fibre coordinate $(y^1, \cdots, y^n)$, but the metric L is not necessary induced from a Riemannian metric on the bace space M. The following theorem is an epoch-making theorem in Finsler geometry.

Theorem 3.2. (Szabo[Sz]) *If (M, L) is a Berwald space, then there exists a Riemannian metric $g = \sum g_{ij}(x) dx^i \otimes dx^j$ on M such that*

$$D = \widetilde{\nabla}^g \tag{3.20}$$

for the Levi-Civita connection ∇^g in (M, g).

3.5 Landsberg spaces

In this section we shall consider the case where the parallel translation P_c is an isometry between the tangential Riemannian spaces. Since the fiber $V_{(x,y)}$ of the vertical bundle V at any point $(x, y) \in TM$ is identified with the tangent space $T_y(T_xM)$, the metric G on V given by (2.11) defines a Riemannian metric G_x on the fiber T_xM. We call such a Riemannian space (T_xM, G_x) a *tangential Riemannian space.*

Definition 3.4. A Finsler manifold (M, L) is called a *Landsberg space* if the parallel translation P_c with respect to the Berwald non-linear connection θ is an isometry between the tangential Riemannian spaces.

From the definition, if c is closed curve with the base point $p \in M$, then the parallel translation $P_c : T_pM \to T_pM$ is a isometric transformation in the Riemannian space (T_pM, G_p). Thus the holonomy group H_p at p in a Landsberg space is a Lie group.

By definition (M, L) is a Landsberg spaces if and only if

$$\mathcal{L}_{X^H} G = 0 \tag{3.21}$$

is satisfied for any vector field X on M. For any vector fields Y, Z on M and their vertical lifts Y^V, Z^V, both of $\mathcal{L}_{X^H} Y^V$ and $\mathcal{L}_{X^H} Z^V$ are vertical vector fields. Then we have

$$\begin{aligned}(\mathcal{L}_{X^H} G)(Y^V, Z^V) &= X^H G(Y^V, Z^V) - G(\mathcal{L}_{X^H} Y^V, Z^V) - G(Y^V, \mathcal{L}_{X^H} Z^V) \\ &= X^H G(Y^V, Z^V) - G(D_{X^H} Y^V, Z^V) - G(Y^V, D_{X^H} Z^V) \\ &= (D_{X^H} G)(Y^V, Z^V).\end{aligned}$$

Therefore (3.21) is equivalent to

$$D_{X^H} G = 0 \tag{3.22}$$

Proposition 3.9. *A Finsler manifold (M, L) is a Landsberg space if and only if the Berwald connection D is compatible with the metric G in horizontal direction.*

The Berwald connection D satisfies (3.4) thus $X^H(L^2) \equiv 0$ for any vector field X on M:

$$\frac{\partial L^2}{\partial x^i} - \sum N_i^l \frac{\partial L^2}{\partial y^l} = 0. \tag{3.23}$$

Here we suppose that (M, L) is a Berwald space. Then the connection coefficients of D are given by the coefficients $\Gamma^i_{jk}(x)$ of a linear connection ∇ in TM. Differentiating (3.23) by y^j and y^k continuously, we have

$$0 = \left(\frac{\partial}{\partial x^i}\right)^H G_{jk} - \sum \Gamma^l_{ij}(x) G_{lk} - \sum \Gamma^l_{ik}(x) G_{jl} = D_{(\partial/\partial x^i)^H} G_{jk}.$$

This shows that if (M, L) is a Berwald space, then the Berwald connection D is compatible with the metric G of V in horizontal direction. Therefore we have

Proposition 3.10. *Any Berwald space (M, L) is a Landsberg space.*

Remark 3.1. As far as the another knows, there is no example of Landsberg space which is not a Berwald space. Still finding an example of non-Berwald Landsberg space is an important open problem in Finsler geometry. □

Let $d\mu$ be a differential n-form on TM defined by

$$d\mu = \sqrt{\det G} \quad dy^1 \wedge \cdots \wedge dy^n. \tag{3.24}$$

The restriction of $d\mu$ to each fiber $V_{(x,y)}$ of the vertical bundle V defines a volume form on $V_{(x,y)}$. For any vector field X on M, the Lie derivative $\mathcal{L}_{X^H} d\mu$ by the horizontal lift X^H is given by

$$\begin{aligned}
&\mathcal{L}_{X^H} d\mu \left(\frac{\partial}{\partial y^1}, \cdots, \frac{\partial}{\partial y^n}\right) \\
&= X^H \left(d\mu\left(\frac{\partial}{\partial y^1}, \cdots, \frac{\partial}{\partial y^n}\right)\right) - \sum d\mu\left(\frac{\partial}{\partial y^1}, \cdots, \left[X^H, \frac{\partial}{\partial y^k}\right], \cdots, \frac{\partial}{\partial y^n}\right) \\
&= \sum X^j(x)\left[\left(\frac{\partial}{\partial x^j}\right)^H \sqrt{\det G} - \sum d\mu\left(\frac{\partial}{\partial y^1}, \cdots, \sum \frac{\partial N_j^m}{\partial y^k}\frac{\partial}{\partial y^m}, \cdots, \frac{\partial}{\partial y^n}\right)\right] \\
&= \sum X^j(x)\left[\left(\frac{\partial}{\partial x^j}\right)^H \sqrt{\det G} - \sum \Gamma^m_{jm}\sqrt{\det G}\right] \\
&= \sum X^j(x)\left[\frac{1}{2}(\det G)^{-1/2}. \left(\frac{\partial}{\partial x^j}\right)^H \det G - \left(\sum \Gamma^m_{jm}\right)\sqrt{\det G}\right].
\end{aligned}$$

If (M, L) is a Landsberg space, then (3.22) implies

$$\left(\frac{\partial}{\partial x^j}\right)^H \sqrt{\det G} = 2 \det G \sum \Gamma^m_{jm}.$$

Therefore we obtain

Proposition 3.11. *If (M, L) is a Landsberg space, then*

$$\mathcal{L}_{X^H} d\mu = 0 \tag{3.25}$$

for any vector field X on M.

Let $\{\varphi_t^H\}$ be the (local) flow of the horizontal lift X^H. For any compact subset $K_0 \subset T_{x_0}M$, we set $K_t = \varphi_t^H(K_0)$. Then the volume $vol(K_t)$ of K_t is given by

$$vol(K_t) = \int_{K_t} d\mu = \int_{\varphi_t^H(K_0)} d\mu = \int_{K_0} \varphi_t^{H^*}(d\mu).$$

If (M, L) is a Landsberg space, then (3.25) implies $\varphi_t^{H^*}(d\mu) = d\mu$, thus $vol(K_t) = vol(K_0)$. In particular, the volume of the indicatrix I_x is independent of the base point $x \in M$.

Proposition 3.12. *For a Landsberg space (M,L), the volume $vol(I_x)$ of the indicatrix I_x is constant.*

3.6 Averaged metrics and connections

Let $I_x = \{y \in T_xM | L(x, y) = 1\}$ be the indicatrix at $x \in M$. For any $y \in I_x$ the identity

$$G_{(x,y)}(\mathcal{E}, \mathcal{E}) = L(x, y)^2 = 1$$

holds, thus the Liouville vector field $\mathcal{E}$ on E is a unit vector field at each point $y \in I_x$.

$$\sum G^{im} \frac{\partial L}{\partial y^m} \left(\frac{\partial}{\partial y^i} \right) = \frac{1}{2L} \sum G^{im} \frac{\partial L^2}{\partial y^m} \left(\frac{\partial}{\partial y^i} \right) = \frac{1}{L} \sum G^{im} G_{lm} y^l \left(\frac{\partial}{\partial y^i} \right) = \frac{1}{L} \mathcal{E}(x, y)$$

also implies that $\mathcal{E}$ is the outward-standing unit normal field of I_x at each point $y \in I_x$. Therefore, if we define $d\mu_I$ by

$$d\mu_I = \iota(\mathcal{E}) d\mu, \tag{3.26}$$

we may consider the restriction of $d\mu_I$ to each fibre as the volume form of I_x.

For the horizontal lift X^H of any vector field X on M, the fact

$$d\pi_{(x,y)}(\mathcal{L}_{X^H} \mathcal{E}) = d\pi_{(x,y)}[X^H, \mathcal{E}] = 0$$

implies the following.

Lemma 3.1. *The Liouville vector field $\mathcal{E}$ is invariant by parallel translation, that is,*

$$\mathcal{L}_{X^H} \mathcal{E} = 0 \tag{3.27}$$

is satisfied for the horizontal lift X^H of any vector field X on M. In particular, if (M, L) is a Landsberg space,

$$\mathcal{L}_{X^H} d\mu_I = 0. \tag{3.28}$$

Let X and Y be vector fields on the base space M. For the respective vertical lifts X^V and Y^V, it is easy to see that the map $g_x : T_xM \otimes T_xM \to \mathbb{R}$ defined by

$$g_x(X, Y) = \frac{1}{vol(I_x)} \int_{I_x} G(X^V, Y^V) d\mu_I \tag{3.29}$$

is an inner product on T_xM. Therefore $g = \{g_x\}$ defines a Riemannian metric on the base space M.

Definition 3.5. ([Ma-Ra-Tr-Ze]) The Riemannian metric g defined by (3.29) is called the *averaged Riemannian metric* on the Finsler manifold (M, L).

Let g be the averaged Riemannian metric on a Finsler manifold (M, L) and X a vector field on M. Then we define a map $\overline{\nabla}_X : T_xM \to T_xM$ so that

$$g(\overline{\nabla}_X Y, Z) = \frac{1}{vol(I_x)} \int_{I_x} G(D_{X^H} Y^V, Z^V) d\mu_I \tag{3.30}$$

is satisfied for any $Y, Z \in T_xM$. We show that $\overline{\nabla} : T_xM \otimes T_xM \ni (Y, Z) \longmapsto \overline{\nabla}_Y Z \in T_xM$ is a linear connection on TM. By definition it is enough to show that $\overline{\nabla}$ satisfies the Leibniz rule. For an arbitrary function $f \in C^\infty(M)$ we have

$$\begin{aligned} g(\overline{\nabla}_X(f \cdot Y), Z) &= \frac{1}{vol(I_x)} \int_{I_x} G(D_{X^H}(f.Y)^V, Z^V) d\mu_I \\ &= \frac{1}{vol(I_x)} \int_{I_x} G(X(f)Y^V + f D_{X^H} Y^V, Z^V) d\mu_I \\ &= \frac{X(f)}{vol(I_x)} \int_{I_x} G(Y^V, Z^V) d\mu_I + \frac{f}{vol(I_x)} \int_{I_x} G(D_{X^H} Y^V, Z^V) d\mu_I \\ &= X(f) g(Y, Z) + f g(\overline{\nabla}_X Y, Z) \\ &= g(X(f) \cdot Y + f \cdot \overline{\nabla}_X Y, Z) \end{aligned}$$

for any vector fields X, Y in M. Therefore we obtain

$$\overline{\nabla}_X(f \cdot Y) = X(f)Y + f \cdot \overline{\nabla}_X Y,$$

thus $\overline{\nabla}$ defined by (3.30) is a linear connection on TM. Moreover we have

Lemma 3.2. *The linear connection $\overline{\nabla}$ defined by* (3.30) *is symmetric.*

Proof. The following identity holds:

$$\begin{aligned} (Dd\pi)(X^H, Y^H) &= D_{X^H} d\pi(Y^H) - D_{Y^H} d\pi(X^H) - d\pi([X^H, Y^H]) \\ &= D_{X^H} Y^V - D_{Y^H} X^V - d\pi([X, Y]^H) \\ &= D_{X^H} Y^V - D_{Y^H} X^V - [X, Y]^V. \end{aligned}$$

Thus, since the Berwald connection D is symmetric, the identity (3.3) implies

$$g(\overline{\nabla}_X Y - \overline{\nabla}_Y X - [X,Y], Z) = \frac{1}{vol(I_x)} \int_{I_x} G(D_{X^H} Y^V - D_{Y^H} X^V - [X,Y]^V, Z^V) d\mu_I = 0$$

for any vector fields X, Y and Z on M. Consequently we have $\overline{\nabla}_X Y - \overline{\nabla}_Y X - [X,Y] = 0$. □

Definition 3.6. (cf. [To-Et]) The symmetric linear connection $\overline{\nabla}$ defined by (3.30) is called the *averaged connection* on the Finsler manifold (M, L).

Then, as a generalization of Theorem 3.2, we have

Theorem 3.3. [Ai3] *If (M, L) is a Landsberg space, then the averaged connection $\overline{\nabla}$ is the Levi-Civita connection of the averaged Riemannian metric g.*

Proof. Let X be an arbitrary vector field on M. We denote by $\{\varphi_t\}$ the 1-parameter family of local transformation group generated by v, and by $\{\varphi_t^H\}$ the horizontal lift of $\{\varphi_t\}$. In Proposition 3.4 we have proved the equation $\varphi_t^H(I_x) = I_{\varphi_t(x)}$.

We assume that (M, L) is a Landsberg space. Then (3.28) shows

$$vol(I_{\varphi_t(x)}) = \int_{\varphi_t^H(I_x)} d\mu_I = \int_{I_x} \varphi_t^{H^*} d\mu_I = \int_{I_x} d\mu_I = vol(I_x)$$

therefore the volume $vol(I_x)$ of indicatrix is constant. Then we have

$$\begin{aligned} X\left(\int_{I_x} f(x,y) d\mu_I\right) &= \frac{d}{dt}\bigg|_{t=0} \left[\int_{I_{\varphi_t(x)}} f(x,y) d\mu_I\right] \\ &= \frac{d}{dt}\bigg|_{t=0} \left[\int_{\varphi_t^H(I_x)} f(x,y) d\mu_I\right] \\ &= \int_{I_x} \frac{d}{dt}\bigg|_{t=0} [\varphi_t^{H*} f] d\mu_I \\ &= \int_{I_x} X^H(f) d\mu_I \end{aligned}$$

for all $X \in \Gamma(TM)$ and $f \in C^\infty(TM)$, where φ_t and φ_t^H denote the 1-parameter family of local transformations generated by X and X^H respectively. Thus we have

$$X(g(Y,Z)) = \frac{1}{vol(I_x)} \int_{I_x} X^H(G(Y^V, Z^V)) d\mu_I.$$

Therefore, from (3.22) we obtain

$$\begin{aligned}
\overline{\nabla}_X g(Y,Z) &= X(g(Y,Z)) - g(\overline{\nabla}_X Y, Z) - g(Y, \overline{\nabla}_X Z) \\
&= \frac{1}{vol(I_x)} \int_{I_x} [X^H(G(Y^V, Z^V)) - G(D_{X^H} Y^V, Z^V) - G(Y^V, D_{X^H} Z^V)] d\mu_I \\
&= \frac{1}{vol(I_x)} \int_{I_x} (D_{X^H} G)(Y^V, Z^V) d\mu_I \\
&= 0.
\end{aligned}$$

Since $\overline{\nabla}$ is symmetric, $\overline{\nabla}$ is the Levi-Civita connection of the averaged Riemannian metric g. □

Suppose that (M, L) is a Berwald space. Since the Berwald connection D is induced from a symmetric linear connection ∇' on TM, that is, $D_{X^H} Y^V = (\nabla'_X Y)^V$, the averaged connection $\overline{\nabla}$ is given by

$$g(\overline{\nabla}_X Y, Z) = \frac{1}{vol(I_x)} \int_{I_x} G((\nabla'_X Y)^V, Z^V) d\mu_I = g(\nabla'_X Y, Z),$$

thus $\overline{\nabla} = \overline{\nabla}'$. Since every Berwald space is a Landsberg space, the averaged connection $\overline{\nabla}$ is the Levi-Civita connection ∇^g of the averaged Riemannian metric g. Consequently Theorem 3.2 is rewritten as follows.

Theorem 3.4. *The Berwald connection D of a Berwald space is induced from the Levi-Civita connection ∇^g of the averaged Riemannian metric g.*

Chapter 4

Rund connections

4.1 Rund connections

The Berwald connection D on (M, L) is the canonical connection defined in Chapter 1, however, D is not necessary metrical with respect to the metric G on V. In this sense the Berwald connection D is somewhat unfortunate. In this section we shall introduce another Finsler connection ∇ on V which satisfies the *almost G-compatibility*. Similarly to the case of D, we also assume that ∇ is flat in the vertical direction.

For this purpose, for every $X \in \Gamma(TM)$, we shall define $P_{X^H} : \Gamma(V) \to \Gamma(V)$ by

$$(D_{X^H}G)(Y^V, Z^V) = 2G(P_{X^H}(Y^V), Z^V) \tag{4.1}$$

for all vector fields Y, Z on M. Then it is easily shown that P_{X^H} is a tensor field, i.e.,

- $P_{X^H}(Y^V + Z^V) = P_{X^H}(Y^V) + P_{X^H}(Z^V)$
- $P_{X^H}(f \cdot Y^V) = f P_{X^H}(Y^V)$ for all $f \in C^\infty(M)$,

i.e., $P_{X^H} \in \mathrm{End}(V)$. Then we can easily show that P_{X^H} is symmetric:

$$G(P_{X^H}(Y^V), Z^V) = G(Y^V, P_{X^H}(Z^V)) \tag{4.2}$$

for all $Y, Z \in \Gamma(TM)$. Therefore (4.1) is written as follows:

$$X^H(G(Y^V, Z^V)) - G(D_{X^H}Y^V + P_{X^H}(Y^V), Z^V) - G(Y^V, D_{X^H}Z^V + P_{X^H}(Z^V)) = 0$$

If we define $\nabla_{X^H} : \Gamma(V) \to \Gamma(V)$ by

$$\nabla_{X^H} Y^V = D_{X^H} Y^V + P_{X^H}(Y^V). \tag{4.3}$$

then ∇_{X^H} is a covariant derivation in V such that

$$\nabla_{X^H} G = 0. \tag{4.4}$$

Since $H = ker(\theta)$ is defined by the Berwald non-linear connection θ, it is natural to assume that ∇ recovers the Berwald non-linear connection θ similarly to the case of the Berwald connection D, namely, we assume

$$\nabla \mathcal{E} = \theta \tag{4.5}$$

Then, from (4.3) we have

Proposition 4.1. *The connection ∇ satisfies* (4.5) *if and only if the tensor field P satisfies*

$$P_{X^H}(\mathcal{E}) = 0 \tag{4.6}$$

for every $X^H \in \Gamma(H)$.

A vector field Y on M is parallel with respect to θ if and only if the vertical lift Y^V along Y is covariantly constant with respect to the Berwald connection D, that is, $D_{X^H}(\mathcal{E} \circ Y) = 0$ for all vector field X on M. Furthermore, since (4.3) and (4.6) imply

$$\nabla_{X^H}(\mathcal{E} \circ Y) = D_{X^H}(\mathcal{E} \circ Y) + P_{X^H}(\mathcal{E} \circ Y) = D_{X^H}(\mathcal{E} \circ Y),$$

we have

Proposition 4.2. *A vector field Y on a Finsler manifold (M, L) is parallel with respect to the Berwald non-linear connection θ if and only if the vertical lift Y^V along Y is covariantly constant with respect to ∇, namely*

$$\nabla_{X^H}(\mathcal{E} \circ Y) = 0 \tag{4.7}$$

for all $X \in \Gamma(TM)$.

Lastly, since D satisfies the symmetry condition (3.3), it is natural to assume that ∇ also satisfies the symmetry condition

$$\nabla d\pi \equiv 0. \tag{4.8}$$

Definition 4.1. Let θ be the Berwald non-linear connection on a Finsler manifold (M, L). A connection $\nabla : \Gamma(V) \to A^1(V)$ is called a *Rund connection* if ∇ satisfies the following conditions.

(1) ∇ associates with θ, i.e., ∇ satisfies (4.5).

(2) ∇ is symmetric, i.e., ∇ satisfies (4.8).

(3) ∇ is almost G-compatible, i.e., ∇ satisfies (4.4).

(4) ∇ is flat in the vertical direction.

Proposition 4.3. *The Rund connection ∇ on a Finsler manifold (M, L) is uniquely determined.*

Since θ is the Berwald non-linear connection, (3.4) implies

$$\nabla_{X^H} L = X^H(L) = 0$$

for any $X \in \Gamma(TM)$. Therefore the Rund connection ∇ is also almost $L - compatible$.

Proposition 4.4. *The connection ∇ satisfies* (4.8) *if and only if the tensor field P satisfies*

$$P_{X^H}(Y^V) = P_{Y^H}(X^V) \tag{4.9}$$

for all vector fields X, Y in M.

Proof. Since (3.3) and (4.8) show that

$$(\nabla d\pi)(X^H, Y^H) = \nabla_{X^H} Y^V - \nabla_{Y^H} X^V - [X, Y]^V = 0$$

and

$$(Dd\pi)(X^H, Y^H) = D_{X^H} Y^V - D_{Y^H} X^V - [X, Y]^V = 0$$

for all $X, Y \in \Gamma(TM)$, we have

$$\begin{aligned} P_{X^H}(Y^V) - P_{Y^H}(X^V) &= \nabla_{X^H} Y^V - D_{X^H} Y^V - (\nabla_{Y^H} X^V - D_{Y^H} X^V) \\ &= \nabla_{X^H} Y^V - \nabla_{Y^H} X^V - [X, Y]^V - (D_{X^H} Y^V - D_{Y^H} X^V - [X, Y]^V) \\ &= 0. \end{aligned}$$

□

The identities (4.2) and (4.9) lead us to

Proposition 4.5. *The covariant derivative $D_{X^H}G$ is totally symmetric, that is,*

$$(D_{X^H}G)(Y^V, Z^V) = (D_{Y^H}G)(Z^V, X^V) = (D_{Z^H}G)(X^V, Y^V) \tag{4.10}$$

for all vector fields X, Y, and Z on M.

Remark 4.1. From (4.10), the covariant derivative DG of the metric G in the horizontal direction is totally symmetric. Hence the pair (G, D) is called a *Finsler-statistical structure* in [Na-Ai]. If we put

$$D_{(\partial/\partial x^k)^H}\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) := G_{ij;k},$$

then (4.10) is written as $G_{ij;k} = G_{jk;i} = G_{ki;j}$.

Let Π^i_j be the connection form of ∇ :

$$\nabla \frac{\partial}{\partial y^j} = \sum \frac{\partial}{\partial y^i} \otimes \Pi^i_j$$

Then the connection form Π^i_j is of the form $\Pi^i_j = \sum \Pi^i_{jk}(x, y)dx^k$, where Π^i_{jk} are local functions satisfying

$$\Pi^i_{jk} = \Pi^i_{kj} \tag{4.11}$$

from (4.8). Furthermore (4.4) implies

$$\left(\frac{\partial}{\partial x^k}\right)^H G_{ij} - \sum G_{rj}\Pi^r_{ik} - \sum G_{ir}\Pi^r_{jk} = 0.$$

Therefore (4.11) implies

$$\Pi^i_{jk} = \frac{1}{2}\sum G^{im}\left[\left(\frac{\partial}{\partial x^k}\right)^H G_{jm} + \left(\frac{\partial}{\partial x^j}\right)^H G_{mk} - \left(\frac{\partial}{\partial x^m}\right)^H G_{jk}\right]. \tag{4.12}$$

The assumption (4.5) means that

$$\sum y^j \Pi^i_{jk} = N^i_k \tag{4.13}$$

for the coefficients N^i_j of the Berwald non-linear connection θ.

From (4.3), the tensor field P_{X^H} is given by

$$P_{X^H}\left(\frac{\partial}{\partial y^j}\right) = \nabla_{X^H}\frac{\partial}{\partial y^j} - D_{X^H}\frac{\partial}{\partial y^j} = \sum \frac{\partial}{\partial y^i}\left[\Pi^i_j(X^H) - \omega^i_j(X^H)\right].$$

Therefore, if we define a horizontal 1-form P^i_j by $P^i_j = \sum(\varPi^i_{jk} - \varGamma^i_{jk})dx^k$, then P_{X^H} is given by

$$P_{X^H}\left(\frac{\partial}{\partial y^j}\right) = \sum P^i_j(X^H)\frac{\partial}{\partial y^i}.$$

4.2 Curvature and torsion of Rund connection

We shall show an expression of the curvature $R_\nabla = \nabla^2$ of the Rund connection ∇ in local coordinates. Since ∇ is also relatively flat in the vertical direction, the curvature R_∇ is also decomposed as $R_\nabla = R^{HH}_\nabla + R^{HV}_\nabla$, where

$$R^{HH}_\nabla(X,Y)Z^V = \nabla_{X^H}\nabla_{Y^H}Z^V - \nabla_{Y^H}\nabla_{X^H}Z^V - \nabla_{[X^H,Y^H]}Z^V$$

and

$$R^{HV}_\nabla(X,Y)Z^V = \nabla_{X^H}\nabla_{Y^V}Z^V - \nabla_{Y^V}\nabla_{X^H}Z^V - \nabla_{[X^H,Y^V]}Z^V$$

for all vector fields X, Y and Z on M. The curvature form $\varOmega_\nabla = d\varPi + \varPi\wedge\varPi$ with respect to $\dfrac{\partial}{\partial y} = \{\partial/\partial y^1, \cdots, \partial/\partial y^n\}$ is also defined by

$$R_\nabla\frac{\partial}{\partial y} = \frac{\partial}{\partial y}\otimes\varOmega_\nabla = \frac{\partial}{\partial y}\otimes(\varOmega^{HH}_\nabla + \varOmega^{HV}_\nabla),$$

where we put

$$\varOmega^{HH}_\nabla = d^H\varPi + \varPi\wedge\varPi \quad \text{and} \quad \varOmega^{HV}_\nabla = d^V\varPi.$$

By direct calculations we have

$$R_\nabla\frac{\partial}{\partial y^j} = \sum\frac{\partial}{\partial y^i}\otimes\left(d\varPi^i_j + \sum\varPi^i_m\wedge\varPi^m_j\right).$$

Then

$$\begin{aligned}
&d\varPi^i_j + \sum\varPi^i_m\wedge\varPi^m_j\\
&= \sum\left[\sum\left(\frac{\partial}{\partial x^l}\right)^H\varPi^i_{jk}dx^l + \sum\frac{\partial\varPi^i_{jk}}{\partial y^l}\theta^l\right]\wedge dx^k + \sum\left(\sum\varPi^i_{mk}dx^k\right)\wedge\left(\sum\varPi^m_{jl}dx^l\right)\\
&= \sum\left[\left(\frac{\partial}{\partial x^l}\right)^H\varPi^i_{jk} - \left(\frac{\partial}{\partial x^k}\right)^H\varPi^i_{jl} + \sum\varPi^i_{mk}\varPi^m_{jl} - \sum\varPi^i_{ml}\varPi^m_{jk}\right]dx^k\wedge dx^l\\
&\quad + \sum\left(-\frac{\partial\varPi^i_{jk}}{\partial y^l}\right)dx^k\wedge\theta^l.
\end{aligned}$$

Therefore R_∇ is given by

$$R_\nabla \frac{\partial}{\partial y^j} = \sum \frac{\partial}{\partial y^i} \otimes \left[\sum_{k<l} R^i_{jkl} dx^k \wedge dx^l + \sum P^i_{jkl} dx^k \wedge \theta^l \right], \tag{4.14}$$

where the coefficients R^i_{jkl} and P^i_{jkl} of Ω_∇ are given by

$$R^i_{jkl} \stackrel{\text{def}}{=} \left(\frac{\partial}{\partial x^l}\right)^H \Pi^i_{jk} - \left(\frac{\partial}{\partial x^k}\right)^H \Pi^i_{jl} + \sum \Pi^i_{mk}\Pi^m_{jl} - \sum \Pi^i_{ml}\Pi^m_{jk} \tag{4.15}$$

and

$$P^i_{jkl} \stackrel{\text{def}}{=} -\frac{\partial \Pi^i_{jk}}{\partial y^l} \tag{4.16}$$

respectively.

Proposition 4.6. *A Finsler manifold (M, L) is a Berwald space if and only if R^{HV}_∇ vanishes identically.*

By the assumption (4.8) the projection $d\pi$ is always parallel with respect to ∇. Then we define the torsion tensor field T_∇ of ∇ similarly to the one T_D of D.

Definition 4.2. The V-valued 2-form T_∇ defined by

$$T_\nabla \stackrel{\text{def}}{=} \nabla\theta \tag{4.17}$$

is called the *torsion* of ∇.

Since ∇ is also flat in the vertical direction and V is integrable, we obtain $T_\nabla(X_V, Y_V) \equiv 0$ for all vector fields X, Y on TM. Therefore T_∇ splits as $T_\nabla = T^{HH}_\nabla + T^{HV}_\nabla$.

First, we have

$$\begin{aligned} T^{HH}_\nabla(X,Y) &:= T_\nabla(X^H, Y^H) \\ &= \nabla_{X^H}\theta(Y^H) - \nabla_{Y^H}\theta(X^H) - \theta([X^H, Y^H]) \\ &= -\theta([X^H, Y^H]) \\ &= d\theta(X^H, Y^H) \\ &= T_D(X^H, Y^H), \end{aligned}$$

that is,

$$T^{HH}_\nabla = T^{HH}_D. \tag{4.18}$$

Second, we have

$$\begin{aligned} T_{\nabla}^{HV}(X,Y) &:= T_{\nabla}(X^H, Y^V) \\ &= \nabla_{X^H}\theta(Y^V) - \nabla_{Y^V}\theta(X^H) - \theta([X^H, Y^V]) \\ &= \nabla_{X^H}Y^V - \theta(\mathcal{L}_{X^H}Y^V) \\ &= \nabla_{X^H}Y^V - D_{X^H}Y^V \\ &= P_{X^H}(Y^V), \end{aligned}$$

therefore we obtain

$$T_{\nabla}^{HV}(X,Y) = P_{X^H}(Y^V). \tag{4.19}$$

With respect to the local frame field $\left\{\frac{\partial}{\partial y^1}, \cdots, \frac{\partial}{\partial y^n}\right\}$ the torsion T_∇ is given by

$$T_{\nabla} = \sum \frac{\partial}{\partial y^i} \otimes \left(d\theta^i + \sum \mathit{\Pi}^i_j \wedge \theta^j \right). \tag{4.20}$$

The torsion form T^i_∇ is given by

$$T^i_{\nabla} = d\theta^i + \sum \omega^i_j \wedge \theta^j + \sum P^i_j \wedge \theta^j = T^i_D + \sum P^i_j \wedge \theta^j.$$

Consequently, from (3.15), the torsion T_∇ of ∇ is given by

$$T_{\nabla} = \sum \frac{\partial}{\partial y^i} \otimes \left(R^i_{jk} dx^j \wedge dx^k \right) + \sum \frac{\partial}{\partial y^i} \otimes \left(P^i_{jk} dx^j \wedge \theta^k \right),$$

where the coefficients R^i_{jk} are given by (3.16) and P^i_{jk} are defined by

$$P^i_{jk} \stackrel{\text{def}}{=} \mathit{\Pi}^i_{jk} - \mathit{\Gamma}^i_{jk} \tag{4.21}$$

respectively.

Proposition 4.7. *A Finsler manifold (M, L) is a Landsberg space if and only if T_{∇}^{HV} vanishes identically.*

4.3 Some identities

We list up some identities concerning R_∇ and T_∇. The definition of T_∇ and the Ricci identity imply $T_\nabla = R_\nabla \mathcal{E} = R_{\nabla}^{HH}\mathcal{E} + R_{\nabla}^{HV}\mathcal{E}$. Thus we have

Proposition 4.8. *The curvature R_∇ and the torsion T_∇ satisfies the relation $T_\nabla = R_\nabla \mathcal{E}$:*

$$R_\nabla^{HH}\mathcal{E} = T_\nabla^{HH} \tag{4.22}$$

and

$$R_\nabla^{HV}\mathcal{E} = T_\nabla^{HV}. \tag{4.23}$$

The second identity and Proposition 4.7 implies the following proposition.

Proposition 4.9. *Every Berwald space is a Landsberg space.*

The symmetry assumption (4.8) and Ricci identity $\nabla^2(d\pi) = R_\nabla \wedge d\pi$ imply $R_\nabla \wedge d\pi = 0$. The LHS of this identity implies

$$\begin{aligned}(R_\nabla \wedge d\pi)(X^H, Y^H, Z^H) &= R_\nabla(X^H, Y^H)d\pi(Z^H) + R_\nabla(Y^H, Z^H)d\pi(X^H) \\ &\quad + R_\nabla(Z^H, X^H)d\pi(Y^H) \\ &= R_\nabla^{HH}(X,Y)Z^V + R_\nabla^{HH}(Y,Z)X^V + R_\nabla^{HH}(Z,X)Y^V\end{aligned}$$

and

$$\begin{aligned}(R_\nabla \wedge d\pi)(X^H, Y^V, Z^H) &= R_\nabla(X^H, Y^V)d\pi(Z^H) + R_\nabla(Y^V, Z^H)d\pi(X^H) \\ &\quad + R_\nabla(Z^H, X^H)d\pi(Y^V) \\ &= R^{HV}(X,Y)Z^V - R_\nabla^{HV}(Z,Y)X^V.\end{aligned}$$

Therefore we obtain the following.

Proposition 4.10. (Bianchi identities) *The horizontal part R_∇^{HH} and the mixed part R_∇^{HV} satisfy the following:*

$$R_\nabla^{HH}(X,Y)Z^V + R_\nabla^{HH}(Y,Z)X^V + R_\nabla^{HH}(Z,X)Y^V \equiv 0 \tag{4.24}$$

and

$$R^{HV}(X,Y)Z^V - R_\nabla^{HV}(Z,Y)X^V \equiv 0. \tag{4.25}$$

The G-compatibility assumption (4.4) gives rise to

$$\begin{aligned}
&[X^H, Y^H]G(Z^V, W^V) \\
&= X^H(Y^H G(Z^V, W^V)) - Y^H(X^H G(Z^V, W^V)) \\
&= X^H(G(\nabla_{Y^H} Z^V, W^V) + G(Z^V, \nabla_{Y^H} W^V)) - Y^H(G(\nabla_{X^H} Z^V, W^V) + G(Z^V, \nabla_{X^H} W^V)) \\
&= G((\nabla_{X^H}\nabla_{Y^H} - \nabla_{Y^H}\nabla_{X^H})Z^V, W^V) + G(Z^V, (\nabla_{X^H}\nabla_{Y^H} - \nabla_{Y^H}\nabla_{X^H})W^V)
\end{aligned}$$

and

$$\begin{aligned}
[X^H, Y^H]G(Z^V, W^V) &= (\nabla_{[X^H, Y^H]})(Z^V, W^V) + G(\nabla_{[X^H, Y^H]} Z^V, W^V) + G(Z^V, \nabla_{[X^H, Y^H]} W^V) \\
&= -2C(T^{HH}_{\nabla}(X, Y), Z^V, W^V) + G(\nabla_{[X^H, Y^H]} Z^V, W^V) \\
&\quad + G(Z^V, \nabla_{[X^H, Y^H]} W^V).
\end{aligned}$$

Therefore we obtain the following.

Proposition 4.11. *The curvature R_{∇} and the torsion T_{∇} satisfy the following:*

$$G(R^{HH}_{\nabla}(X, Y)Z^V, W^V) + G(R^{HH}_{\nabla}(X, Y)W^V, Z^V) + 2C(T^{HH}_{\nabla}(X, Y), Z^V, W^V) = 0 \tag{4.26}$$

$$\begin{aligned}
G(R^{HV}_{\nabla}(X, Y)Z^V, W^V) + G(Z^V, R^{HV}_{\nabla}(X, Y)W^V) + 2(\nabla_{X^H} C)(Y^V, Z^V, W^V) \\
+ 2C(T^{HV}_{\nabla}(X, Y), Z^V, W^V) = 0
\end{aligned} \tag{4.27}$$

Proof. The second identity is obtained by direct computations using the almost G-compatibility assumption (4.4). □

We suppose that $R^{HH}_{\nabla} = 0$. The identity (4.22) implies $T^{HH}_{\nabla} = 0$ and the horizontal subbundle H is integrable. Then Proposition 4.2 guarantees the existence of a parallel vector field $\zeta : U \longrightarrow TM|_U$. Since ζ is parallel, we have

$$d\zeta(X) = X^H \in H_{\zeta(x)} \tag{4.28}$$

at each $\zeta(x)$ for all $X \in \Gamma(TM)$. Then we define a local metric g^{ζ} on the open set U by

$$g^{\zeta}_x(Y, Z) := G_{\zeta(x)}(\mathcal{E} \circ Y, \mathcal{E} \circ Z) = G_{\zeta(x)}(Y^V, Z^V)$$

for all $Y, Z \in \Gamma(TM)$, where the superscript "V" denotes the vertical lift along ζ, e.g., $Y^V(x) = (\mathcal{E} \circ Y)_{\zeta(x)}$. We define a linear connection ∇^ζ on $TM|_U$ by

$$(\nabla^\zeta_X Y)^V := \nabla_{d\zeta(X)} Y^V = \nabla_{X^H} Y^V$$

for all $X, Y \in \Gamma(TM)$. Then ∇^ζ is flat. In fact, since $[X^H, Y^H] = [X, Y]^H$ because of the integrability of H, we obtain

$$\begin{aligned}((\nabla^\zeta \circ \nabla^\zeta Z(X, Y))^V &= (\nabla^\zeta_X \nabla^\zeta_Y Z)^V - (\nabla^\zeta_Y \nabla^\zeta_X Z)^V - (\nabla^\zeta_{[X,Y]} Z)^V \\ &= \nabla_{X^H} \nabla_{Y^H} Z^V - \nabla_{Y^H} \nabla_{X^H} Z^V - \nabla_{[X,Y]^H} Z^V \\ &= \nabla_{X^H} \nabla_{Y^H} Z^V - \nabla_{Y^H} \nabla_{X^H} Z^V - \nabla_{[X^H,Y^H]} Z^V \\ &= R^{HH}_\nabla (X, Y) Z^V \\ &= 0\end{aligned}$$

for all $X, Y, Z \in \Gamma(TM)$. Therefore ∇^ζ is a flat connection on $TM|_U$. Furthermore

$$\begin{aligned}(\nabla^\zeta_X g^\zeta)(Y, Z) &= X g^\zeta(Y, Z) - g^\zeta(\nabla^\zeta_X Y, Z) - g^\zeta(Y, \nabla^\zeta_X Z) \\ &= d\zeta(X) G(Y^V, Z^V) - G(\nabla_{d\zeta(X)} Y^V, Z^V) - G(Y^V, \nabla_{d\zeta(X)} Z^V) \\ &= X^H G(Y^V, Z^V) - G(\nabla_{X^H} Y^V, Z^V) - G(Y^V, \nabla_{X^H} Z^V) \\ &= (\nabla_{X^H} G)(Y^V, Z^V) \\ &= 0\end{aligned}$$

implies that the metric g^ζ is a flat Riemannian metric on U, since ∇^ζ is torsion-free. Therefore M is locally Euclidean.

Theorem 4.1. *If a smooth manifold M admits a Finsler metric satisfying $R^{HH}_\nabla \equiv 0$, then M is locally Euclidean.*

The curvature R_D and the torsion T_D satisfy some important identities. For later convenience, we are concerned with the following identity.

$$R^{HV}_D (X, Y) Z^V = R^{HV}_\nabla (X, Y) Z^V + (\nabla_{X^H} T^{HV}_\nabla)(Y, Z). \tag{4.29}$$

4.4 Variational formulae in Finsler manifolds

The contains of this section refer from [Ai-Ko]. Let (M, L) be a Finsler manifold with the Rund connection ∇. The canonical lift $\widetilde{c} : I \to TM$ of a regular oriented curve $c = c(t)$ in M is defined by $\widetilde{c}(t) = (c(t), \dot{c}(t))$. The velocity field of $\widetilde{c}$ is given by

$$\frac{d\widetilde{c}}{dt} = \left(\frac{dc}{dt}\right)^H + \nabla_{(dc/dt)^H}\left(\mathcal{E} \circ \frac{dc}{dt}\right) = \dot{c}^H + \nabla_{(dc/dt)^H}\left(\mathcal{E} \circ \frac{dc}{dt}\right),$$

where $\dot{c}^H$ and $\dot{c}^V$ are vertical and horizontal lifts of the velocity field $\dot{c} = dc/dt$ along $\widetilde{c}$ respectively. Here the second term in RHD of the above is given by

$$\nabla_{\dot{c}^H}\dot{c}^V = \sum \left[\frac{d^2x^i}{dt^2} + \sum \mathit{\Pi}^i_{jk}(x, \dot{x})\frac{dx^j}{dt}\frac{dx^k}{dt}\right]\frac{\partial}{\partial y^i}.$$

Definition 4.3. A regular oriented curve $c : I \to M$ is called a *path* if

$$\frac{d\widetilde{c}}{dt} = \dot{c}^H \tag{4.30}$$

or equivalently

$$\frac{d^2x^i}{dt^2} + \sum \mathit{\Pi}^i_{jk}(x, \dot{x})\frac{dx^j}{dt}\frac{dx^k}{dt} = 0 \tag{4.31}$$

is satisfied. In particular, if the parameter t is a normal parameter of c with respect to L, then $c = c(t)$ is called a *geodesic* in (M, L).

Let $c = c(t)$ be a smooth curve in M.

Definition 4.4. A section $\mathcal{Z} \in \mathit{\Gamma}(V)$ is said to be parallel along its natural lifts $\widetilde{c}$ if $\mathcal{Z}$ satisfies $\widetilde{c}^*\nabla\mathcal{Z} = 0$. Especially the vertical lift X^V of s vector field X on M is parallel along $\widetilde{c}$ if

$$\widetilde{c}^*(\nabla X^V) = 0 \tag{4.32}$$

is satisfied.

Since X^V is the vertical lift of $X \in \mathit{\Gamma}(TM)$, the covariant derivative of X^V in the vertical direction vanishes identically. Thus we have

$$\widetilde{c}^*(\nabla X^V)\left(\frac{d}{dt}\right) = \nabla X^V\left(\frac{d\widetilde{c}}{dt}\right) = \nabla X^V(\dot{c}^H + \nabla_{\dot{c}^H}\dot{c}^V) = \nabla_{\dot{c}^H}X^V.$$

Hence the vertical lift X^V along $\widetilde{c}$ is parallel if and only if

$$\nabla_{\dot{c}^H}X^V = \sum \left[\frac{dX^i}{dt} + \sum X^j \mathit{\Pi}^i_{jk}(x, \dot{x})\frac{dx^k}{dt}\right]\frac{\partial}{\partial x^i} = 0,$$

namely

$$\frac{dX^i}{dt} + \sum \Pi^i_{jk}(x, \dot{x}) X^j \frac{dx^k}{dt} = 0 \tag{4.33}$$

is satisfied.

Remark 4.2. The parallelism along a curve c in M and the one along the natural lift $\dot{c}$ in TM must be distinguished strictly. A vector field X on M is parallel along a curve c in M if $\widetilde{c}_X^* \theta = 0$ is satisfied, where $\widetilde{c}_X$ is the lift of c defined by $\widetilde{c}_X(t) = (c(t), (X \circ c)(t))$. Since ∇ satisfies (4.5), this definition can be written as

$$(\widetilde{c}_X^* \theta)\left(\frac{d}{dt}\right) = \theta\left(\frac{d\widetilde{c}_X}{dt}\right) = \theta(\dot{c}^H + \mathcal{L}_{\dot{c}^H}(\mathcal{E} \circ X)) = \nabla_{\dot{c}^H}(\mathcal{E} \circ X) = 0.$$

Therefore X is parallel along c if and only if X satisfies

$$\frac{dX^i}{dt} + \sum \Pi^i_{jk}(x, X) X^j \frac{dx^k}{dt} = 0. \tag{4.34}$$

If ∇ is induced from a linear connection on TM, e.g., if (M, L) is a Berwald space, the vertical lift X^V is parallel along $\widetilde{c}$ if and only if X is so along c.

In the sequel, we use the notation $\nabla_t X^V$ instead of $\widetilde{c}^*(\nabla X^V)$ for any $X \in \Gamma(TM)$ and its vertical lift X^V along $\widetilde{c}$:

$$\nabla_t X^V = \nabla_{\dot{c}^H} X^V.$$

Let X and Y be vector fields along a path c in (M, L). Then, if c is a path in (M, L), then we have

$$\frac{d}{dt} G(X^V, Y^V) = G(\nabla_t X^V, Y^V) + G(X^V, \nabla_t Y^V), \tag{4.35}$$

since $d\widetilde{c}/dt$ is horizontal. Hence we have

Proposition 4.12. *Let c be a path in a Finsler manifold (M, L). If X^V and Y^V are parallel along $\widetilde{c}$, then the inner product $G(X^V, Y^V)$ is constant along $\widetilde{c}$.*

A regular oriented curve $\gamma(t) = (x^i(t))$ with normal parameter t is a geodesic if and only if (4.31) is satisfied. Proposition 4.12 shows that, if γ is a geodesic, the tangent vector $\dot{\gamma}$ has a constant norm and γ has constant speed. In the sequel, we always assume that the parameter of a geodesic is normal unless otherwise stated.

Let $\gamma_X : I \to M$ be a geodesic with initial point $x = \gamma_X(0)$ and the initial direction $X = \dot{\gamma}_X(0)$, where the parameter t is, of course, normal. We shall define the *exponential map* exp by $\exp(x, X) = \gamma_X(1)$ if $X \neq 0$ and $\exp(x, 0) = x$. The restriction of exp to $\mathcal{D} \cap T_x M$ is denoted by $\exp_x$. The restricted exponential map $\exp_p$ maps the rays through the origin $0 \in T_xM$ to the unique geodesics through the point x in a sufficiently small ball $B_x(r) = \{X \in T_xM | \|v\| < r\}$.

The exponential map exp is defined on an open neighborhood $\mathcal{D}$ of the zero section $o(M)$ of TM, and exp is C^∞-class away from $o(M)$. Furthermore exp is C^1-class at $o(M)$, and its derivative at $o(M)$ is the identity map. By a result due to Akbar-Zaedah, the map exp is C^2-class at $o(M)$ if and only if (M, L) is a Berwald space (see [Ba-Ch-Sh]).

For each $X \in T_xM$, the *radial geodesic* γ_X is given by $\gamma_X(t) = \exp_x(tX)$ for all $t \in I$ such that either side is defined. This geodesic segment γ_X has the tangent vector field $\dot\gamma_X$ with $\dot\gamma_X(0) = X$. Since $\nabla_t(\dot\gamma_X)^V = 0$, the identity (4.35) implies that $\|\dot\gamma_X\|^2 = G((\dot\gamma_X)^V, (\dot\gamma_X)^V)$ is constant along γ_X, thus $\|\dot\gamma_X(t)\| = \|\dot\gamma_X(0)\| = \|X\|$. Consequently we have $\int_0^1 \|\dot\gamma_X(t)\| dt = \|X\|$.

4.4.1 The first variation of arc length and geodesics

We shall show the first variation formula in Finsler manifolds. For this end we introduce some definitions.

Let $c = c(t) \in \Gamma(p, q)$ be a regular oriented curve with unit speed, that is, $\|\dot c(t)\| = 1$. Then a *variation* of c is a family $\{c_s\}$ of oriented curve $c_s(t)$ parameterized by $s \in (-\varepsilon, \varepsilon)$ such that $c_0(t) = c(t)$ for all $t \in I$. A variation Γ_c is said to be *proper* if it fixes the end points, that is, $c_s(0) = p$ and $c_s(1) = q$. We suppose that the map $\Gamma_c : (-\varepsilon, \varepsilon) \times I \to M$ defined by $\Gamma_c(s, t) = c_s(t)$ is smooth. For the variational problem of arc length, it is enough to assume that Γ_c is piecewise differentiable with respect to the parameter t (cf. [Ma], Chapter VIII). However, we shall assume the smoothness of Γ_c for the sake of simplicity of discussions.

By the assumption the map Γ_c satisfies $\Gamma_c(0, t) = c(t), p = \Gamma_c(s, 0)$ and $q = \Gamma_c(s, 1)$. Setting s =constant for each $s \in (-\varepsilon, \varepsilon)$, the parameterized curve $c_s : I \to M$ defined by $c_s(t) = \Gamma_c(s, t)$ is called a *s-curve*, while the parameterized curve $c_t(s) = \Gamma_c(s, t)$ is a *t-curve* which is a transversal curve to c. In local coordinates, we set $\Gamma_c(s, t) = (x^1(s, t), \cdots, x^n(s, t))$. We denote by $\mathcal{S} = \partial c_t/\partial s$ and $\mathcal{T} = \partial c_s/\partial t$ the tangent vector fields of *t-curve* and *s-curve* respectively:

$$\mathcal{S} = \sum \frac{\partial x^i}{\partial s} \frac{\partial}{\partial x^i}, \qquad \mathcal{T} = \sum \frac{\partial x^i}{\partial t} \frac{\partial}{\partial x^i}.$$

In particular, the vector field $\Theta(t)$ along c defined by

$$\Theta(t) = \left(\frac{\partial c_t}{\partial s}\right)_{(0,t)} = \mathcal{S}(0, t)$$

is called the *variational field* induced from Γ_c. If Γ_c satisfies $c_s(0) = c(0) = p$ and $c_s(1) = c(1) = q$ for all $s \in (-\varepsilon, \varepsilon)$, then the variational field Θ is *proper*, that is, Θ satisfies $\Theta(0) = \Theta(1) = 0$.

We are always concerned with the variation Γ_c whose variational field Θ is independent of the tangent vector $\dot{c}$ at least one point on c. Let $\Theta = \Theta(t)$ be any vector field along a regular oriented curve $c = c(t)$. Then there exists a variation Γ_c which induces Θ as its variational field. In fact, if we take $\Gamma_c(s,t) = \exp(s\Theta(t))$, then $\Gamma_c : (-\varepsilon, \varepsilon) \times I \to M$ is a variation of c with variational field Θ.

Lemma 4.1. *Let Θ be any vector field along c. Then Θ is a variational field of some variation Γ_c of c. If Θ is proper, then Θ is the variational field induced from a certain proper variation Γ_c.*

Let $\mathcal{S}^V$ and $\mathcal{T}^V$ be the vertical lifts of $\mathcal{S}$ and $\mathcal{T}$ along the canonical lift $\widetilde{c}_s$ of s-curve c_s respectively:

$$\mathcal{S}^V(\widetilde{c}_s(t)) = \sum \frac{\partial x^i}{\partial s}\left(\frac{\partial}{\partial y^i}\right)_{\widetilde{c}_s}, \qquad \mathcal{T}^V(\widetilde{c}_s(t)) = \sum \frac{\partial x^i}{\partial t}\left(\frac{\partial}{\partial y^i}\right)_{\widetilde{c}_s}.$$

Lemma 4.2. *Let $\Gamma_c : (-\varepsilon, \varepsilon) \times I \to M$ be a variation. Then we have*

$$\nabla_{\mathcal{S}^H}\mathcal{T}^V = \nabla_{\mathcal{T}^H}\mathcal{S}^V \tag{4.36}$$

along $\widetilde{c}_s = (c_s(t), \dot{c}_s(t)))$.

Proof. Along the curve $\widetilde{c}_s$ we have

$$\nabla_{\mathcal{S}^H}\mathcal{T}^V = \sum\left[\frac{\partial^2 x^i}{\partial s \partial t} + \sum \Pi^i_{jk}(\widetilde{c}_s(t))\frac{\partial x^j}{\partial t}\frac{\partial x^k}{\partial s}\right]\left(\frac{\partial}{\partial y^i}\right)_{\widetilde{c}_s}$$

and

$$\nabla_{\mathcal{T}^H}\mathcal{S}^V = \sum\left[\frac{\partial^2 x^i}{\partial t \partial s} + \sum \Pi^i_{jk}(\widetilde{c}_s(t))\frac{\partial x^j}{\partial s}\frac{\partial x^k}{\partial t}\right]\left(\frac{\partial}{\partial y^i}\right)_{\widetilde{c}_s}$$

Then (4.36) is obtained from (4.11). □

Let Γ_c be a proper variation of a regular oriented curve $c \in \Gamma(p,q)$. We compute the first variation of the length functional $\mathcal{F}_L(c_s)$. Since $G(\mathcal{T}^V, \mathcal{T}^V) = L(c_s(t), \dot{c}_s(t))^2 = \mathcal{F}_L(c_s)^2$, we have

$$\frac{d}{ds}\mathcal{F}_L(c_s) = \frac{1}{2}\int_0^1 \frac{1}{\|\mathcal{T}\|}\frac{\partial G(\mathcal{T}^V, \mathcal{T}^V)}{\partial s}dt.$$

Furthermore, (4.35) and (4.36) imply

$$\frac{1}{\|\mathcal{T}\|}\frac{\partial G(\mathcal{T}^V, \mathcal{T}^V)}{\partial s} = \frac{2}{\|\mathcal{T}\|}G(\nabla_{\mathcal{S}^H}\mathcal{T}^V, \mathcal{T}^V) = \frac{2}{\|\mathcal{T}\|}G(\nabla_{\mathcal{T}^H}\mathcal{S}^V, \mathcal{T}^V) \tag{4.37}$$

along $\widetilde{c}_s$. Consequently we have

$$\frac{1}{\|\mathcal{T}\|}\frac{\partial G(\mathcal{T}^V,\mathcal{T}^V)}{\partial s} = \frac{2}{\|\mathcal{T}\|}\left[\frac{d}{dt}G(\mathcal{S}^V,\mathcal{T}^V) - G(\mathcal{S}^V,\nabla_{\mathcal{T}^H}\mathcal{T}^V)\right],$$

which gives us

$$\frac{d}{ds}\mathcal{F}_L(c_s) = \int_0^1 \frac{1}{\|\mathcal{T}\|}\left[\frac{d}{dt}G(\mathcal{S}^V,\mathcal{T}^V) - G(\mathcal{S}^V,\nabla_{\mathcal{T}^H}\mathcal{T}^V)\right]dt.$$

Evaluating $s = 0$, $\|\mathcal{T}\|_{s=0} = \|\dot{c}(t)\| = 1$ derives the following:

Proposition 4.13. (First Variation Formula) *Let $c : I \to M$ be a regular oriented curve and Γ_c a proper variation of c. Then*

$$\left.\frac{d}{ds}\right|_{s=0}\mathcal{F}_L(c_s) = -\int_0^1 G(\Theta^V,\nabla_t\dot{c}^V)dt, \tag{4.38}$$

where Θ is the variational field of Γ_c.

A regular oriented curve c is said to be a *stationary* point of the functional $\mathcal{F}_L$ if

$$(d\mathcal{F}_L(c_s)/ds)_{s=0} = 0$$

for any proper variation Γ_c. If a regular oriented curve $c : I \to M$ is a geodesic, then c satisfies (2.3), thus c is a stationary point of $\mathcal{F}_L$ from (2.6).

Conversely we suppose that c is a stationary point of the functional $\mathcal{F}_L$. Since the condition

$$\left(\frac{\mathcal{F}_L(c_s)}{ds}\right)_{s=0} = 0$$

is satisfied for any variational field Θ along c, we take $\Theta(t) = \varphi(t)\nabla_t\dot{c}$ for a smooth function φ satisfying $\varphi(0) = \varphi(1) = 0$ and $\varphi > 0$ elsewhere. Then, since Θ is proper and from (4.38), we have

$$\left.\frac{d}{ds}\right|_{s=0}\mathcal{F}_L(c_s) = -\int_0^1 \varphi(t)\|\nabla_t\dot{c}^V\|^2dt,$$

which implies $\nabla_t\dot{c}^V = 0$ on I.

Proposition 4.14. *A regular oriented curve γ in a Finsler manifold (M, L) is a stationary point of the functional $\mathcal{F}_L$ if and only if γ is a geodesic from p to q.*

From Proposition 2.2 we have

Theorem 4.2. *Every $\mathcal{F}_L$-minimizing curve γ in (M, L) is a geodesic if γ is regular.*

The converse of this theorem is also true.

Theorem 4.3. *Every geodesic γ in a Finsler manifold (M, L) is locally $\mathcal{F}_L$-minimizing.*

This theorem is proved by using the Gauss lemma. We define the geodesic ball $\mathcal{B}_x(r)$ centered at $x \in M$ of radius r by $\mathcal{B}_x(r) = \exp(B(r))$ for the tangential ball $B_x(r) = \{X \in T_xM | \|X\| < r\}$. Let $S_x(r) = \{X \in T_xM | \|X\| = r\}$ be the tangent sphere. Then the set $\mathcal{S}_x(r) = \exp(S_x(r))$ is called the *geodesic sphere* at x of radius r. Then the Gauss lemma is stated as follows.

Lemma 4.3. (The Gauss lemma) *The radial geodesic γ_X is orthogonal to the geodesic sphere $\mathcal{S}_x(r)$ at $x \in M$.*

For a proof of Theorem 4.3, we need more technical preliminaries, but we omit them here. For the complete proof, see [Ba-Ch-Sh] or [Ch-Ch-La].

4.4.2 The Jacobi fields and conjugate points

A variation $\varGamma_\gamma = \varGamma_\gamma(s, t)$ of a geodesic γ is said to be a *geodesic variation* if each s-curve γ_s is also a geodesic. Since each s-curve γ_s is a geodesic, we have $\nabla_{\mathcal{T}^H} \mathcal{T}^V = 0$.

Let X be a vector field along γ_s. Then, since $[\mathcal{S}, \mathcal{T}] = 0$, we have

$$\nabla_{\mathcal{S}^H} \nabla_{\mathcal{T}^H} X^V - \nabla_{\mathcal{T}^H} \nabla_{\mathcal{S}^H} X^V = R^{HH}(\mathcal{S}, \mathcal{T}) X^V \tag{4.39}$$

along γ_s. From this equation, we get the so-called the *Jacobi equation*.

Proposition 4.15. (The Jacobi Equation) *Let γ be a geodesic and $\varTheta$ the variational field of a geodesic variation $\varGamma_\gamma$ of γ in a Finsler manifold (M, L). Then $\varTheta$ satisfies*

$$\nabla_t \nabla_t \varTheta^V + R^{HH}(\varTheta, \dot{\gamma}) \dot{\gamma}^V = 0. \tag{4.40}$$

Proof. Since each curve γ_s is a geodesic, we have $\nabla_t \mathcal{T}^V = 0$, and this yields $\nabla_s \nabla_t \mathcal{T}^V = 0$. On the other hand the symmetric property (4.36) and the equation (4.39) imply

$$\nabla_{\mathcal{S}^H} \nabla_{\mathcal{T}^H} \mathcal{T}^V = \nabla_{\mathcal{T}^H} \nabla_{\mathcal{S}^H} \mathcal{T}^V + R^{HH}(\mathcal{S}, \mathcal{T}) \mathcal{T}^V = \nabla_{\mathcal{T}^H} \nabla_{\mathcal{T}^H} \mathcal{S}^V + R^{HH}(\mathcal{S}, \mathcal{T}) \mathcal{T}^V$$

along γ_s. Since $\mathcal{S}(0, t) = V(t)$ and $\mathcal{T}(0, t) = \dot{\gamma}$, we obtain (4.40). □

Definition 4.5. Let (M, L) be a Finsler manifold. The differential equation (4.40) is called the *Jacobi equation*. A vector field J along a geodesic γ satisfying (4.40):

$$\nabla_t \nabla_t J^V + R^{HH}(J, \dot\gamma)\dot\gamma^V = 0 \tag{4.41}$$

is called a *Jacobi field* in (M, L).

By definition the variational field Θ of a geodesic variation of a geodesic γ is a Jacobi field. Conversely every Jacobi field along a geodesic γ is the variational field of some geodesic variation of γ. The differential equation (4.41) is linear and of second order, we have $2n$ linearly independent solutions. Therefore, along any geodesic γ, the set of Jacobi field is a $2n$-dimensional vector space.

Definition 4.6. Let $\gamma \in \Gamma(p, q)$ be a geodesic segment in M. Then q is said to be conjugate along γ if there exists a Jacobi field $J(\neq 0)$ along γ such that J vanishes at p and q.

For $X \in T_pM$, we set $q = \exp_p X$. For an arbitrary $Y \in T_pM$, we shall compute the differential $(\exp_p)_* Y$ at X:

$$(\exp_p)_* Y = \left.\frac{d}{ds}\right|_{s=0} (\exp_p)(X + sY).$$

To compute $(\exp_p)_*$, we define a geodesic variation Γ_γ of γ_X by $\Gamma_\gamma(s, t) = \exp_p t(X + sY)$. The variational field $J = \partial\Gamma_\gamma/\partial s$ is a Jacobi field along γ_X, and we have $J(1) = (\exp_p)_* Y$. The conjugate points are the image of the singularities by the exponential mapping.

Proposition 4.16. *Let $\gamma_X(t) = \exp_p(tX) \quad (t \in I)$ be the radial geodesic for $X \in T_xM$. Then $\exp_p$ is a local diffeomorphism if and only if $q = \exp_p X$ is not conjugate to p along γ_X.*

4.4.3 The second variational formula and index form

Let γ be a geodesic with unit speed. We shall compute the second variation of the length functional $\mathcal{F}_L$. We shall compute

$$\left.\frac{d^2}{ds^2}\right|_{s=0} \mathcal{F}_L(\gamma_s) = \int_0^1 \left[\frac{\partial}{\partial s}\frac{G(\nabla_{\mathcal{T}^H}\mathcal{S}^V, \mathcal{T}^V)}{\|\mathcal{T}\|}\right]_{s=0} dt.$$

Differentiating this with respect to s, we have

$$\frac{\partial}{\partial s}\frac{G(\nabla_{\mathcal{T}^H}\mathcal{S}^V, \mathcal{T}^V)}{\|\mathcal{T}\|} = -\frac{1}{\|\mathcal{T}\|^2}\frac{\partial\|\mathcal{T}\|}{\partial s}G(\nabla_{\mathcal{T}^H}\mathcal{S}^V, \mathcal{T}^V) + \frac{1}{\|\mathcal{T}\|}\frac{\partial}{\partial s}G(\nabla_{\mathcal{T}^H}\mathcal{S}^V, \mathcal{T}^V).$$

From (2.2) and (2.4), we get

$$\frac{\partial\|\mathcal{T}\|}{\partial s} = \frac{1}{\|\mathcal{T}\|} G(\nabla_{\mathcal{T}^H}\mathcal{S}^V, \mathcal{T}^V).$$

Furthermore

$$\begin{aligned}\frac{\partial}{\partial s} G(\nabla_{\mathcal{T}^H}\mathcal{S}^V, \mathcal{T}^V) &= G(\nabla_{\mathcal{S}^H}\nabla_{\mathcal{T}^H}\mathcal{S}^V, \mathcal{T}^V) + G(\nabla_{\mathcal{T}^H}\mathcal{S}^V, \nabla_{\mathcal{S}^H}\mathcal{T}^V) \\ &= G(\nabla_{\mathcal{T}^H}\nabla_{\mathcal{S}^H}\mathcal{S}^V + R^{HH}(\mathcal{S},\mathcal{T})\mathcal{S}^V, \mathcal{T}^V) + G(\nabla_{\mathcal{T}^H}\mathcal{S}^V, \nabla_{\mathcal{T}^H}\mathcal{S}^V).\end{aligned}$$

Consequently we have

$$\begin{aligned}&\frac{d^2\mathcal{F}_L(\gamma_s)}{ds^2} \\ &= \int_0^1 \frac{1}{\|\mathcal{T}\|}\left[G(\nabla_{\mathcal{T}^H}\nabla_{\mathcal{S}^H}\mathcal{S}^V + R^{HH}(\mathcal{S},\mathcal{T})\mathcal{S}^V, \mathcal{T}^V) + \|\nabla_{\mathcal{T}^H}\mathcal{S}^V\|^2 - \frac{G(\nabla_{\mathcal{T}^H}\mathcal{S}^V, \mathcal{T}^V)^2}{\|\mathcal{T}\|^2}\right] dt\end{aligned}$$

along γ_s. Since $\nabla_{\mathcal{T}^H}\mathcal{T}^V = 0$ and $\Theta(0) = \Theta(1) = 0$ imply

$$\begin{aligned}\int_0^1 [G(\nabla_{\mathcal{T}^H}\nabla_{\mathcal{S}^H}\mathcal{S}^V, \mathcal{T}^V)]_{s=0} dt &= \int_0^1 \left[\frac{\partial}{\partial t} G(\nabla_{\mathcal{S}^H}\mathcal{S}^V, \mathcal{T}^V)\right]_{s=0} dt \\ &= G(\nabla_{\Theta^H}\Theta^V, \dot\gamma^V)_{t=1} - G(\nabla_{\Theta^H}\Theta^V, \dot\gamma^V)_{t=0} \\ &= 0,\end{aligned}$$

we have

$$\left.\frac{d^2}{ds^2}\right|_{s=0} L_F(\gamma_s) = \int_0^1 [G(R^{HH}(\Theta,\dot\gamma)\Theta^V, \dot\gamma^V) + \|\nabla_t\Theta^V\|^2 - G(\nabla_t\Theta^V, \dot\gamma^V)^2] dt. \tag{4.42}$$

Let $\Theta^V_\top = G(\Theta^V, \dot\gamma^V)\dot\gamma^V$ be the tangential part of Θ^V. We also denote by $\Theta^V_\perp$ the normal part of Θ^V, that is, $\Theta^V_\perp = \Theta^V - \Theta^V_\top$. Then $\nabla_t\dot\gamma^V = 0$ implies $\nabla_t\Theta^V_\top = \nabla_t(G(\Theta^V, \dot\gamma^V)\dot\gamma^V) = (\nabla_t\Theta^V)^\top$ and $\nabla_t\Theta^V_\perp = \nabla_t\Theta^V - \nabla_t\Theta^V_\top$. Hence we obtain

$$\|\nabla_t\Theta^V\|^2 = \|\nabla_t\Theta^V_\top\|^2 + \|\nabla_t\Theta^V_\perp\|^2 = G(\nabla_t\Theta^V, \dot\gamma^V)^2 + \|\nabla_t\Theta^V_\perp\|^2.$$

Then, since $G(R^{HH}(\dot\gamma,\dot\gamma)\bullet,\bullet) = 0$ and $C(\dot\gamma,\bullet,\bullet) = 0$ along γ, we have

$$G(R^{HH}(\bullet,\bullet)\dot\gamma^V, \dot\gamma^V) = 0.$$

Hence we get

$$G(R^{HH}(\Theta,\dot\gamma)\Theta^V,\dot\gamma^V) = G(R^{HH}(\Theta_\perp,\dot\gamma)\Theta^V_\perp,\dot\gamma^V).$$

Consequently we obtain the second variation formula of L_F.

Proposition 4.17. (Second Variation Formula) *Let $\gamma : I \to M$ be any geodesic with unit speed, Γ_γ a proper variation of γ and Θ its variation field. Then*

$$\left.\frac{d^2}{ds^2}\right|_{s=0} L_F(\gamma_s) = \int_0^1 [G(R^{HH}(\Theta_\perp,\dot\gamma)\Theta^V_\perp,\dot\gamma^V) + \|\nabla_t\Theta^V_\perp\|^2]dt, \tag{4.43}$$

where $\Theta_\perp$ is the normal part of Θ.

The Bianchi identity implies

$$G(R^{HH}(\Theta_\perp,\dot\gamma)\Theta^V_\perp,\dot\gamma^V) = -G(R^{HH}(\Theta_\perp,\dot\gamma)\dot\gamma^V,\Theta^V_\perp)$$

along γ, and since $\Theta_\perp$ is normal to $\widetilde{\gamma}$, we obtain

$$G(R^{HH}(\Theta^V_\perp,\dot\gamma)\Theta^V_\perp,\dot\gamma^V) = -\|\Theta^V_\perp\|^2 K(\Theta^V_\perp)$$

for the flag curvature K. Hence the second variation formula has the form

$$\left.\frac{d^2}{ds^2}\right|_{s=0} L_F(\gamma_s) = \int_0^1 [\|\nabla_t\Theta^V_\perp\|^2 - \|\Theta^V_\perp\|^2 K(\Theta^V_\perp)]dt. \tag{4.44}$$

Proposition 4.18. *Let (M, F) be a Finsler manifold with a non-positive flag curvature K. Then the second variation of any geodesic satisfies*

$$\left.\frac{d^2}{ds^2}\right|_{s=0} L_F(\gamma_s) > 0.$$

Proof. The assumption $K \le 0$ induces

$$\|\nabla_t\Theta^V_\perp\|^2 - \|\Theta^V_\perp\|^2 K(\Theta^V_\perp) \ge 0.$$

Hence $(d^2L_F(\gamma_s)/ds^2)_{s=0} \ge 0$. if $(d^2L_F(\gamma_s)/ds^2)_{s=0} = 0$, then we have $\|\nabla_t\Theta^V_\perp\|^2 = 0$, so $\nabla_t\Theta^V_\perp = 0$. Then, since V is proper, we have $\Theta_\perp = 0$, implies that the variational field Θ^V has the form $\Theta^V = \Theta^V_\top = \varphi(t)\dot\gamma^V(t)$ for some function φ on each point of γ. However, this is a contradiction to the assumption that Θ is independent of $\dot\gamma$ at least one point on γ. □

We define the *index form* on a Finsler manifold (M, F). Let γ be a unit speed geodesic in (M, F). We set

$$I(X,Y) = \int_0^1 [G(R^{HH}(X,\dot{\gamma})Y^V,\dot{\gamma}^V) + G(\nabla_t X^V, \nabla_t Y^V)]dt, \tag{4.45}$$

for normal proper vector fields X, Y along γ. The index form I is a symmetric bi-linear form on the space of normal proper vector fields. In fact, the Bianchi identity implies

$$G(R^{HH}(X,\dot{\gamma})Y^V,\dot{\gamma}^V) + G(R^{HH}(\dot{\gamma},Y)X^V,\dot{\gamma}^V) + G(R^{HH}(Y,X)\dot{\gamma}^V,\dot{\gamma}^V) = 0.$$

Since the last term on the left hand side vanishes, we have

$$G(R^{HH}(X,\dot{\gamma})Y^V,\dot{\gamma}^V) = -G(R^{HH}(\dot{\gamma},Y)X^V,\dot{\gamma}^V) = G(R^{HH}(Y,\dot{\gamma})X^V,\dot{\gamma}^V)$$

along γ. Thus I is a symmetric bi-linear form: $I(X,Y) = I(Y,X)$.

Since $G(R^{HH}(X,\dot{\gamma})Y^V,\dot{\gamma}^V) = -G(R^{HH}(X,\dot{\gamma})\dot{\gamma}^V, Y^V)$ along γ, if X and Y are proper, we have

$$\int_0^1 G(\nabla_t X^V, \nabla_t Y^V) = -\int_0^1 G(\nabla_t \nabla_t X^V, Y^V),$$

which implies

$$I(X,Y) = -\int_0^1 [G(\nabla_t \nabla_t X^V - R^{HH}(X,\dot{\gamma})\dot{\gamma}^V, Y^V)]dt. \tag{4.46}$$

By the definition of I and (2.13), the second variation of L_F of the unit speed geodesic is given by $I(X,X)$, and it can be thought as the Hessian of the length functional L_F. Thus, if γ is minimizing, then $I(X,X) \geq 0$ for any proper normal vector field X along γ. The following is a generalization of the well-known theorem in Riemannian geometry which shows that no geodesics is minimizing, passing its first conjugate point.

Theorem 4.4. *If $\gamma \in \Gamma(p,q)$ is a geodesic segment in a Finsler manifold (M,F) such that γ has an interior conjugate point to p, then there exists a proper normal vector field X along γ such that $I(X,X) < 0$. In particular, γ is not minimizing.*

We also consider the completeness of Finsler manifolds.

Definition 4.7. A Finsler manifold (M,F) is said to be *geodesically complete* if the exponential mapping $\exp_x$ is defined on the whole of T_xM for every point $x \in M$.

We denote by $E(p,\delta)$ the subset of the closure $\overline{B(p,\delta)}$ consisting of the points joined by a minimal geodesic to p. Then, if (M,F) is geodesically complete, the following three conditions are mutually equivalent.

(1) $E(p, \delta)$ is compact,

(2) $E(p, \delta) = \overline{B(p, \delta)}$ for all $\delta > 0$,

(3) any ordered two points in M are joined by a minimal geodesic.

We shall introduce another completeness of Finsler manifolds.

Definition 4.8. A sequence $\{p_m\}$ of points in (M, F) is called a Cauchy sequence, if for any $\varepsilon > 0$ there exists an integer N such that $d_F(p_i, p_j) < \varepsilon \quad (i, j > N)$. Then (M, F) is said to be **metrically complete** if any Cauchy sequence in M converges.

The following theorem is a natural generalization in Riemannian geometry.

Theorem 4.5. (Hopf-Rinow Theorem) *Let (M, F) be a connected Finsler manifold. Then the following three conditions are mutually equivalent.*

(1) *(M, F) is geodesically complete,*

(2) *(M, F) is metrically complete with respect to the distance d_f,*

(3) *Any bounded closed subset of M is compact.*

Chapter 5

Geometry of conformal Finsler manifolds

5.1 Conformal class of Finsler metrics

Let M be an n-dimensional smooth connected manifold with a Riemannian metric g. In this section, a linear connection ∇ is said to be *conformal* if the parallel transport with respect to ∇ preserves angles but not the metric g. Thus ∇ is conformal if and only if there exists a one-form $w(g)$ such that

$$\nabla g = 2w(g) \otimes g. \tag{5.1}$$

Let $\widetilde{g} = e^{2\sigma} g$ denote a conformal deformation of g by any smooth function $\sigma \in C^\infty(M)$. If ∇ is also conformal with respect to $\widetilde{g}$, the relation $\nabla \widetilde{g} = 2w(\widetilde{g}) \otimes \widetilde{g}$ implies

$$w(\widetilde{g}) = w(g) + d\sigma. \tag{5.2}$$

In the sequel we shall denote by c the conformal class of g, that is, $c = \{e^\sigma g | \sigma \in C^\infty(M)\}$. Denoting by $\Lambda^1(M)$ the $C^\infty(M)$-module of one-forms on M, a *Weyl structure* on (M, c) is a map $w : c \to \Lambda^1(M)$ satisfying (5.2). The triplet (M, c, w) is called a *Weyl manifold.*

Theorem 5.1. [Fo] *Let (M, c, w) be a Weyl manifold. Then there exists a unique torsion free linear connection ∇ satisfying* (5.1).

Definition 5.1. The linear connection ∇ is called the Weyl connection of (M, c, w).

The form $w_g := w(g)$ in (5.1) depends on $g \in c$, however the exterior derivative dw_g is independent of the choice of $g \in c$. Thus we define $W \in \Lambda^2(M)$ by $W = dw_g$ for any $g \in c$.

Then we say that the Weyl structure w is *closed* if $W = 0$. If w_g is closed, then we may write $w_g = d\sigma_U$ for a local function $\sigma_U = \sigma_U(x)$ defined on an open subset $U \subset M$. Thus, from (5.1), we have

$$\nabla_X(e^{-2\sigma_U} g) = 2\{-d\sigma_U(X) + w_g(X)\}e^{-2\sigma_U} g = 0. \tag{5.3}$$

Therefore the Weyl connection ∇ is the Levi-Civita connection of a local Riemannian metric $e^{-2\sigma_U} g$ if $W = 0$.

Let $L(M)$ be the frame bundle over M with the structure group $GL(n, \mathbb{R})$, and $\mathbb{L} = L(M) \times_\rho \mathbb{R}$ the density line bundle over (M, g), where ρ is the representation of $GL(n, \mathbb{R})$ defined by $\rho : GL(n, \mathbb{R}) \ni g_{UV} \longmapsto |\det g_{UV}| \in GL(1, \mathbb{R})$. Then $\mathbb{L}$ is a trivial line bundle even if M is orientable. We can define an inner product μ_g on $\mathbb{L}$ by the determinant $\det g$, and any inner product $\widetilde{\mu}$ on $\mathbb{L}$ is written as $\widetilde{\mu} = f\mu_g$ for a positive $f \in C^\infty(M)$. If we set $f = e^{2n\sigma}$ for $\sigma \in C^\infty(M)$, $\widetilde{\mu}$ is written as $\widetilde{\mu} = e^{2n\sigma}\mu_g = \mu_{\widetilde{g}}$ for the conformal deformation $\widetilde{g} = e^{2\sigma} g$ of g.

By taking the trace of connection forms, the Levi-Civita connection ∇^g of (M, g) induces a flat connection $\nabla^{\mathbb{L},g}$ on $\mathbb{L}$ such that $\nabla^{\mathbb{L},g}\mu_g = 0$. Since connections on $\mathbb{L}$ form an affine space modeled on $\Lambda^1(M)$, any connection $\nabla^{\mathbb{L}}$ on $\mathbb{L}$ may be written in the form

$$\nabla^{\mathbb{L}}_X = \nabla^{\mathbb{L},g}_X + \beta_g(X)id, \tag{5.4}$$

where id is the identity morphism of $\mathbb{L}$ and $\beta_g \in \Lambda^1(M)$ is determined by $\nabla^{\mathbb{L}}_X\mu_g = \beta_g(X)\mu_g$. For a conformal deformation $\widetilde{g} = e^{2\sigma} g$ of g, the corresponding one $\beta_{\widetilde{g}}$ is given by

$$\beta_{\widetilde{g}} = \beta_g + n d\sigma. \tag{5.5}$$

Hence, defining $w_g \in \Lambda^1(M)$ by $w(g) = w_g := \beta_g/n$, we obtain a Weyl structure w on (M, c).

Conversely, any Weyl structure w on (M, c) determines a connection $\nabla^{\mathbb{L}}$ on $\mathbb{L}$ by $\nabla^{\mathbb{L}} = \nabla^{\mathbb{L},g} + nw_g \otimes id$ for any $g \in c$. Since the curvature of $\nabla^{\mathbb{L}}$ is given by $ndw_g \otimes id$, the Weyl structure w is closed if and only if the corresponding connection $\nabla^{\mathbb{L}}$ on $\mathbb{L}$ is flat.

Because of $V \cong \pi^*TM$, the vertical sub-bundle V is associated with the pull-back $\pi^*L(M)$ of the frame bundle $L(M)$ over M. We denote by $\widetilde{\mathbb{L}} := \pi^*\mathbb{L}$ the pull-back of the density bundle $\mathbb{L}$ over M. We can define an inner product μ_G on $\widetilde{\mathbb{L}}$ by

$$\mu_G\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) = \det G,$$

where $\dfrac{\partial}{\partial y} = \dfrac{\partial}{\partial y^1} \wedge \cdots \wedge \dfrac{\partial}{\partial y^n}$ denotes the natural local frame field for $\widetilde{\mathbb{L}}$.

The Rund connection (H, ∇) induces a Finsler connection (H, ∇^R) on $\widetilde{\mathbb{L}}$ by the trace of connection coefficients, namely,

$$\nabla^R_{(\partial/\partial x^k)^H} \frac{\partial}{\partial y} = \sum \Pi^m_{mk} \frac{\partial}{\partial y},$$

where

$$\Pi^m_{mk} = \left(\frac{\partial}{\partial x^k}\right)^H \log\sqrt{\det G}.$$

By definition (H, ∇^R) is flat in the vertical direction. Then

$$\begin{aligned}\left(\nabla^R_{(\partial/\partial x^k)^H} \mu_G\right)\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) &= \left(\frac{\partial}{\partial x^k}\right)^H \det G - \mu_G\left(\nabla^R_{(\partial/\partial x^k)^H} \frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) - \mu_G\left(\frac{\partial}{\partial y}, \nabla^R_{(\partial/\partial x^k)^H} \frac{\partial}{\partial y}\right) \\ &= \left(\frac{\partial}{\partial x^k}\right)^H \det G - 2\left(\sum \Pi^m_{mk}\right)\det G \\ &= 0.\end{aligned}$$

Hence ∇^R is always compatible with the metric μ_G in the horizontal direction H :

$$\nabla^R_{(\partial/\partial x^k)^H} \mu_G \equiv 0. \tag{5.6}$$

Further the covariant derivative of μ_G in the vertical direction, i.e.,

$$\left(\nabla^R_{\partial/\partial y^k} \mu_G\right)(S, S) = \frac{\partial(\det G)}{\partial y^k}$$

and Deicke's theorem [De] shows that $\nabla^R \mu_G = 0$ if and only if (M, L) is a Riemannian manifold.

Theorem 5.2. *The connection (H, ∇^R) is compatible with the metric μ_G if and only if (M, L) is a Riemannian manifold.*

This theorem is also true for the connection (H, ∇^B) on $\widetilde{\mathbb{L}}$ induced from (H, D). We suppose that ∇^B is compatible wih the metric μ_G in the horizontal direction H, namely, we assume that (H, ∇^B) satisfies

$$\nabla^B_{(\partial/\partial x^k)^H} \mu_G = 0.$$

By easy computations, this assumption is equivalent to

$$\sum \Gamma^m_{mk} = \left(\frac{\partial}{\partial x^k}\right)^H \log\sqrt{\det G} = \sum \Pi^m_{mk}. \tag{5.7}$$

If (M, L) is a Landsberg space, then this condition is satisfied, but not vise-versa in general.

Definition 5.2. ([Mo]) A Finsler manifold (M, L) is said to be a *weak Landsberg space* if $\nabla^B = \nabla^R$.

Let $d\mu = \sqrt{\det G}\, dy^1 \wedge \cdots \wedge dy^n$ denotes the volume form on each Riemannian space (T_xM, G_x). We may consider $d\mu$ as a section of the dual $\widetilde{\mathbb{L}}^*$, namely, the Riemannian density of (T_xM, G_x) (see [La]). If we use the same notation ∇^B for the induced connection on $\widetilde{\mathbb{L}}^*$, the condition (5.7) is equivalent to $\nabla^B_{X^H} d\mu = 0$. Thus (M, L) is weak Landsberg space if and only if

$$\mathcal{L}_{X^H} d\mu = 0 \tag{5.8}$$

is satisfied.

Remark 5.1. In a complex Finsler manifold there exists a non-linear connection H satisfying (5.8) (see [Ha-Ai]).

Theorem 5.3. *A Finsler manifold is a weak Landsberg space if and only if its the Berwald non-linear connection H preserves the density $d\mu$. In a weak Landsberg space, the volume of any compact subset in each fibre is preserved by the Berwald non-linear connection H.*

Since (H, D) satisfies the metrical condition (3.4), the indicatrix I_x defined by L is preserved by the Berwald non-linear connection H. Hence, in a weak Landsberg space, the volume of indicatrix is constant. Such a space plays an important role in [Ba-Ch].

5.2 Finsler-Weyl connections and Wagner connections

In this section we shall extend the notion of Weyl structures to the category of Finsler geometry.

Suppose that the Berwald connection (H, D) of (M, L) is conformal, namely, we suppose that (H, D) satisfies

$$D_{X^H} G = 2\alpha(X)G,$$

for any vector field X on M and for some $\alpha \in \Lambda^1(M)$. Then, since the deflection of (H, D) vanishes, i.e., $D_{X^H}\mathcal{E} = 0$, the identity (3.1) concludes $\mathcal{L}_{X^H} L = \alpha(X)L$. Hence (3.4) implies $\alpha = 0$, that is, (M, L) is a Landsberg space. Due to [Ma2] and [Ki], a Randers metric L is a Landsberg metric if and only if L is a Berwald metric. These facts are also true if we use the the Rund connection (H, ∇) in stead of (H, D)

Proposition 5.1. *If the Berwald connection (H, D) or the Rund connection (H, ∇) is conformal, then (M, L) is a Landsberg space. In particular, for any Randers space (M, L), if its Berwald connection (H, D) or its Rund connection (H, ∇) is conformal, then (M, L) is a Berwald space.*

We shall show that for any Finsler manifold (M, L) there exists a conformal Finsler connection $(\mathcal{H}, \mathcal{D})$.

Proposition 5.2. *Let (M, L) be a Finsler manifold. For any $\alpha \in \Lambda^1(M)$ there exists a unique non-linear connection $\mathcal{H}$ and a Finsler connection $(\mathcal{H}, \mathcal{D})$ satisfying the following conditions for any vector fields X, Y on M :*

(1) *$(\mathcal{H}, \mathcal{D})$ is conformal, i.e.,*

$$\mathcal{D}_{X^{\mathcal{H}}} G = 2\alpha(X) G. \tag{5.9}$$

(2) *$(\mathcal{H}, \mathcal{D})$ is symmetric, i.e.,*

$$\mathcal{D}_{X^{\mathcal{H}}} Y^V - \mathcal{D}_{Y^{\mathcal{H}}} X^V - [X, Y]^V = 0. \tag{5.10}$$

(3) *The deflection $(\mathcal{H}, \mathcal{D})$ vanishes, i.e.,*

$$\mathcal{D}_{X^{\mathcal{H}}} \mathcal{E} = 0. \tag{5.11}$$

Let $\mathcal{N}^i_k$ be the coefficient of $\mathcal{H}$, i.e.,

$$\left(\frac{\partial}{\partial x^k}\right)^{\mathcal{H}} = \frac{\partial}{\partial x^k} - \sum \mathcal{N}^i_k \frac{\partial}{\partial y^i},$$

and let $\mathcal{K}^i_{jk}$ be the coefficients of $(\mathcal{H}, \mathcal{D})$:

$$\mathcal{D}_{(\partial/\partial x^k)^{\mathcal{H}}} \frac{\partial}{\partial y^j} = \sum \mathcal{K}^i_{jk} \frac{\partial}{\partial y^i}.$$

If we put $\alpha = \sum \alpha_k dx^k$, the connection coefficients $\mathcal{K}^i_{jk}$ of $(\mathcal{H}, \mathcal{D})$ is given by

$$\mathcal{K}^i_{jk} = \frac{1}{2} \sum G^{ir} \left[\left(\frac{\partial}{\partial x^k}\right)^{\mathcal{H}} G_{jr} + \left(\frac{\partial}{\partial x^j}\right)^{\mathcal{H}} G_{rk} - \left(\frac{\partial}{\partial x^r}\right)^{\mathcal{H}} G_{jk} \right] - \alpha_j \delta^i_k - \alpha_k \delta^i_j + \alpha^i G_{jk}, \tag{5.12}$$

where we put $\alpha^i = \sum G^{ik} \alpha_k$, and the coefficients $\mathcal{N}^i_k$ are given by

$$\mathcal{N}^i_k = -\sum \mathcal{N}^m_l y^l G^{ir} C_{rkm} + \sum \left\{ {i \atop l\, k} \right\} y^l - \left(\sum \alpha_l y^l\right) \delta^i_k - \alpha_k y^i + \left(\sum G_{lk} y^l\right) \alpha^i$$

with

$$\sum \mathcal{N}^m_l y^l = \sum \left\{ {m \atop k\, l} \right\} y^k y^l + 2 \left(\sum \alpha_l y^l\right) y^m - L^2 \alpha^m$$

and $C_{rkm} = (\partial G_{rk}/\partial y^m)/2$.

In the case of $\alpha = 0$, the Finsler connection $(\mathcal{H}, \mathcal{D})$ in Proposition 5.2 is just the Rund connection (H, ∇) of (M, L). If α is closed, the assumption (5.9) is written as $\mathcal{D}_{X^{\mathcal{H}}}\left(e^{2\sigma_U} G\right) = 0$ for some $\sigma_U \in C^\infty(U)$. Hence $(\mathcal{H}, \mathcal{D})$ is the Rund connection of a local Finsler metric $e^{\sigma_U} L$.

From [Br] two Finsler metrics L and $\widetilde{L}$ are said to be *conformally equivalent* if there exists a smooth function $\sigma \in C^\infty(M)$ such that $\widetilde{L} = e^\sigma L$. We denote by $\mathcal{C}$ the conformal equivalence class of Finsler metrics on M. The pair $(M, \mathcal{C})$ is called a *conformal Finsler manifold*. Any conformal deformation $L \to \widetilde{L} = e^\sigma L$ induces the conformal deformation $G \to \widetilde{G} = e^{2\sigma} G$ of the metric on V. Suppose that the Finsler connection $(\mathcal{H}, \mathcal{D})$ obtained by Proposition 5.2 is also conformal with respect to $\widetilde{G} = e^{2\sigma} G$. Then $\mathcal{D}_{X^{\mathcal{H}}} \widetilde{G} = 2\widetilde{\alpha}(X)\widetilde{G}$ implies $\widetilde{\alpha} = \alpha + d\sigma$. Thus it is reasonable to call a map $\alpha : \mathcal{C} \ni L \to \alpha(L) \in \Lambda^1(M)$ a *Finsler-Weyl structure* if α satisfies

$$\alpha(\widetilde{L}) = \alpha(L) + d\sigma. \tag{5.13}$$

We call the triplet $(M, \mathcal{C}, \alpha)$ a *Finsler-Weyl manifold.*

Definition 5.3. [Ai1] The connection $(\mathcal{H}, \mathcal{D})$ is called the *Finsler-Weyl connection* of $(M, \mathcal{C}, \alpha)$.

Remark 5.2. In [Ko] a Finsler-Weyl structure is also defined by assuming that $\mathcal{H}$ is conformal with respect to the function L, i.e.,

$$\mathcal{L}_{X^{\mathcal{H}}} L = \alpha(X) L, \tag{5.14}$$

for any vector field X on M. Using (3.1) and (5.11), we can easily show that (5.9) implies (5.13). Thus our notion of Finsler-Weyl structure is stronger than that in [Ko].

Denoting by $\alpha_L = \sum \alpha_j dx^j$ the one-form $\alpha_L := \alpha(L)$, (5.9) is written as

$$\frac{\partial G_{ij}}{\partial x^k} - \sum \mathcal{N}^l_k \frac{\partial G_{ij}}{\partial y^l} - \sum G_{lj} \mathcal{K}^l_{ik} - \sum G_{il} \mathcal{K}^l_{jk} = 2\alpha_k G_{ij}, \tag{5.15}$$

and the assumption (5.10) and (5.11) are described as $\mathcal{K}^i_{jk} = \mathcal{K}^i_{kj}$ and $\mathcal{N}^l_k = \sum y^j \mathcal{K}^i_{jk}$ respectively. Since G_{ij} are homogeneous of degree zero with respect to the variables $y^1, \cdots, y^n$, (5.9) implies

$$\frac{\partial G_{ij}}{\partial x^k} - \sum \overline{\mathcal{N}}^l_k \frac{\partial G_{ij}}{\partial y^l} - \sum G_{lj} \overline{\mathcal{K}}^l_{ik} - \sum G_{il} \overline{\mathcal{K}}^l_{jk} = 0, \tag{5.16}$$

where we put $\overline{\mathcal{N}}^l_k = \mathcal{N}^l_k + \alpha_k y^l$ and $\overline{\mathcal{K}}^i_{jk} = \mathcal{K}^i_{jk} + \alpha_k \delta^i_j$. These functions define a non-linear connection $\overline{\mathcal{H}}$ and a Finsler connection $(\overline{\mathcal{H}}, \overline{\mathcal{D}})$ which is *semi-symmetric*, i.e.,

$$\overline{\mathcal{D}}_{X^{\overline{\mathcal{H}}}} Y^V - \overline{\mathcal{D}}_{Y^{\overline{\mathcal{H}}}} X^V - [X,Y]^V = \alpha_L(X) Y^V - \alpha_L(Y) X^V. \tag{5.17}$$

Furthermore, $(\overline{\mathcal{H}}, \overline{\mathcal{D}})$ satisfies

$$\overline{\mathcal{D}}_{X^{\overline{\mathcal{H}}}} G = 0, \tag{5.18}$$

and

$$\overline{\mathcal{D}}_{X^{\overline{\mathcal{H}}}} \mathcal{E} = 0. \tag{5.19}$$

Definition 5.4. ([Ha-Ic]) The Finsler connection $(\overline{\mathcal{H}}, \overline{\mathcal{D}})$ is called the *Wagner connection* of $(M, \mathcal{C}, \alpha)$.

Therefore the Finsler-Weyl connection $(\mathcal{H}, \mathcal{D})$ of $(M, \mathcal{C}, \alpha)$ determines the Wagner connection $(\overline{\mathcal{H}}, \overline{\mathcal{D}})$ and vise-versa.

Remark 5.3. A Finsler manifold (M, L) is called a *Wagner space* if its Wagner connection $(\overline{\mathcal{H}}, \overline{\mathcal{D}})$ is induced from a linear connection $\overline{\nabla}$ on TM, i.e., $\overline{\mathcal{D}}_{X^{\overline{\mathcal{H}}}} Y^V = (\overline{\nabla}_X Y)^V$. If (M, L) is a Wagner space and α_L is closed, then (M, L) is locally conformal to a Berwald space [Ha-Ic].

5.3 Averaged Riemannian metrics and connections

Let I_x be the indicatrix at $x \in M$ with the volume form $d\mu_I$ defined by (3.26). The volume $v_L(x)$ of I_x is defined by

$$v_L(x) := \int_{I_x} d\mu_I.$$

The averaged Riemannian metric of L is a Riemannian metric g in M defined by (3.29):

$$g(X,Y) = \frac{1}{v_L(x)} \int_{I_x} G(X^V, Y^V) d\mu_I \tag{5.20}$$

for any vector fields X, Y on M.

Let $\widetilde{L} = e^\sigma L$ be a conformal deformation of a Finsler metric L. The indicatrix $\widetilde{I}_x$ at $x \in M$ with respect to $\widetilde{L}$ is given by $\widetilde{I}_x = e^{-\sigma} I_x$, and the volume form $d\mu_{\widetilde{I}}$ on $\widetilde{I}_x$ is given by

$$d\mu_{\widetilde{I}} = \sum (-1)^{i-1} \sqrt{\det \widetilde{G}}\, w^i dw^1 \wedge \cdots \wedge \check{dw}^i \wedge \cdots \wedge dw^n$$

at $w = (w^1, \cdots, w^n) \in \widetilde{I}_x$, where $\widetilde{G} = e^{2\sigma} G$ is the metric on V defined by $\widetilde{L}$. For the diffeomorphism $\psi : I_x \ni (x,y) \longmapsto \psi(x,y) = (x, e^{-\sigma}y) \in \widetilde{I}_x$, we obtain

$$\begin{aligned}
\psi^*(d\mu_{\widetilde{I}}) &= \sum(-1)^{i-1} e^{n\sigma} \sqrt{\det G \circ \psi}\,(e^{-n\sigma} y^i)\, dy^1 \wedge \cdots \wedge d\check{y}^i \wedge \cdots \wedge dy^n \\
&= \sum(-1)^{i-1} \sqrt{\det G}\, y^i\, dy^1 \wedge \cdots \wedge d\check{y}^i \wedge \cdots \wedge dy^n \\
&= d\mu_I,
\end{aligned}$$

which implies

$$v_{\widetilde{L}}(x) = \int_{\widetilde{I}_x} d\mu_{\widetilde{I}} = \int_{I_x} \psi^*(d\mu_{\widetilde{I}}) = \int_{I_x} d\mu_I = v_L(x).$$

Thus, in the sequel we use the notation $v(x)$ instead of $v_L(x)$ for the volume of the indicatrix I_x of any $L \in \mathcal{C}$. The averaged Riemannian metric $\widetilde{g}$ of $\widetilde{L}$ is given by

$$\begin{aligned}
\widetilde{g}(X,Y) &= \frac{1}{v(x)} \int_{\widetilde{I}_x} \widetilde{G}(X^V, Y^V) d\mu_{\widetilde{I}} \\
&= \frac{1}{v(x)} \int_{I_x} e^{2\sigma} (G \circ \psi)(X^V, Y^V)(\psi^* d\mu_{\widetilde{I}}) \\
&= \frac{e^{2\sigma}}{v(x)} \int_{I_x} G(X^V, Y^V) d\mu_I \\
&= e^{2\sigma} g(X,Y).
\end{aligned}$$

Hence $\widetilde{g}$ is given by the conformal deformation

$$\widetilde{g} = e^{2\sigma} g \tag{5.21}$$

of the averaged Riemannian metric g of L.

Theorem 5.4. *Let $\mathcal{C}$ be a conformal class of Finsler metrics on M, and let G be the metric on V determined by any $L \in \mathcal{C}$. Then, by averaging each metric G by* (3.29)*, the class $\mathcal{C}$ determines a conformal class c of Riemannian metrics on M.*

$$\begin{array}{ccc}
\mathcal{C} \ni L & \xrightarrow{\text{conf.}} & \widetilde{L} \in \mathcal{C} \\
\Big\downarrow \text{av} & \circlearrowright & \Big\downarrow \text{av} \\
c \ni g & \xrightarrow[\text{conf.}]{} & \widetilde{g} \in c
\end{array}$$

Let $(\overline{\mathcal{H}}, \overline{\mathcal{D}})$ be the Wagner connection of a Finsler-Weyl manifold $(M, \mathcal{C}, \alpha)$. Then the averaged connection of $(\overline{\mathcal{H}}, \overline{\mathcal{D}})$ is a linear connection $\overline{\nabla}$ on TM defined by

$$g(\overline{\nabla}_X Y, Z) = \frac{1}{v(x)} \int_{I_x} G(\overline{\mathcal{D}}_{X^{\overline{\mathcal{H}}}} Y^V, Z^V) d\mu_I \tag{5.22}$$

for any vector fields X, Y and Z on M, where g is the averaged Riemannian metric of L.

The properties (5.18) and (5.19) of $(\overline{\mathcal{H}}, \overline{\mathcal{D}})$ lead us to

$$\mathcal{L}_{X^{\overline{\mathcal{H}}}} L = 0. \tag{5.23}$$

Hence the parallel displacement with respect to Wagner non-linear connection $\overline{\mathcal{H}}$ preserves every indicatrix, i.e.,

$$I_{\varphi_t(x)} = \varphi_t^{\overline{\mathcal{H}}}(I_x),$$

where φ_t and $\varphi_t^{\overline{\mathcal{H}}}$ denote the flows generated by X and its horizontal lift $X^{\overline{\mathcal{H}}}$ respectively. Therefore we have

$$X\left(\int_{I_x} f d\mu_I\right) = \int_{I_x} \left\{X^{\overline{\mathcal{H}}}(f) d\mu_I + f \mathcal{L}_{X^{\overline{\mathcal{H}}}} d\mu_I\right\} \tag{5.24}$$

for any $f \in C^\infty(M)$.

Now we suppose that Wagner non-linear connection $\overline{\mathcal{H}}$ preserves the density $d\mu$:

$$\mathcal{L}_{X^{\overline{\mathcal{H}}}} d\mu = 0. \tag{5.25}$$

This assumption and $\mathcal{L}_{X^{\overline{\mathcal{H}}}} \mathcal{E} = 0$ lead us to $\mathcal{L}_{X^{\overline{\mathcal{H}}}} d\mu_I = 0$, and thus (5.24) implies that the volume function $v(x)$ is constant. If we normalize $\mathcal{C}$ so that $v(x) = 1$, then (5.18) and (5.25) lead us to

$$\begin{aligned}
&(\overline{\nabla}_X g)(Y, Z) \\
&= Xg(Y, Z) - g(\overline{\nabla}_X Y, Z) - g(Y, \overline{\nabla}_X Z) \\
&= X\left(\int_{I_x} G(Y^V, Z^V) d\mu_I\right) - \int_{I_x} G(\overline{\mathcal{D}}_{X^{\overline{\mathcal{H}}}} Y^V, Z^V) d\mu_I - \int_{I_x} G(Y^V, \overline{\mathcal{D}}_{X^{\overline{\mathcal{H}}}} Z^V) d\mu_I \\
&= \int_{I_x} (\overline{\mathcal{D}}_{X^{\overline{\mathcal{H}}}} G)(Y^V, Z^V) d\mu_I + \int_{I_x} G(Y^V, Z^V) \mathcal{L}_{X^{\overline{\mathcal{H}}}}(d\mu_I) \\
&= 0,
\end{aligned}$$

which shows that $\overline{\nabla}$ is compatible with g. Furthermore, from (5.17),

$$\begin{aligned} g(\overline{\nabla}_X Y - \overline{\nabla}_Y X - [X,Y], Z) &= \int_{I_x} G(\overline{\mathcal{D}}_{X^{\overline{\mathcal{H}}}} Y^V - \overline{\mathcal{D}}_{Y^{\overline{\mathcal{H}}}} X^V - [X,Y]^V, Z^V) d\mu_I \\ &= \int_{I_x} G(\alpha_L(X) Y^V - \alpha_L(Y) X^V - [X,Y]^V, Z^V) d\mu_I \\ &= g(\alpha_L(X)Y - \alpha_L(Y)X, Z), \end{aligned}$$

leads us to

$$\overline{\nabla}_X Y - \overline{\nabla}_Y X - [X,Y] = \alpha_L(X)Y - \alpha_L(Y)X. \tag{5.26}$$

Hence $\overline{\nabla}$ is semi-symmetric, that is, $\overline{\nabla}$ is the so-called Lyra connection of (M,c). Furthermore the connection ∇ defined by

$$\nabla_X Y = \overline{\nabla}_X Y - \alpha_L(X) Y \tag{5.27}$$

is symmetric, and ∇ satisfies

$$\nabla_X g = 2\alpha_L(X) g. \tag{5.28}$$

Consequently the Finsler-Weyl structure α of $(M,\mathcal{C})$ is a Weyl structure of (M,c), and ∇ is the Weyl connection of (M,c,α).

Theorem 5.5. *Let $(M,\mathcal{C},\alpha)$ be a Finsler-Weyl manifold and let (M,c,α) be the Weyl manifold determined by $(M,\mathcal{C},\alpha)$. Suppose that the Wagner non-linear connection $\overline{\mathcal{H}}$ of $(M,\mathcal{C},\alpha)$ preserves the density $d\mu$. Then the averaged connection $\overline{\nabla}$ of the Wagner connection $(\overline{\mathcal{H}},\overline{\mathcal{D}})$ is the Lyra connection of (M,c,α) and the deformed connection ∇ defined by* (5.27) *is the Weyl connection of (M,c,α).*

5.4 Conformal flatness of Finsler metrics

A Finsler manifold (M,L) is said to be *flat* or *locally Minkowski* if there exists a coordinate system $\{U,(x^i)\}_{1\le i\le n}$ on M such that L is independent of $x \in M$.

Definition 5.5. A Finsler manifold (M,L) is said to be *conformally flat* if, for each point $x \in M$, there exists a neighborhood U of x and a function σ_U on U such that $L_U = e^{\sigma_U(x)} L(x,y)$ is a flat Finsler metric in U.

Let $(\mathcal{H},\mathcal{D})$ be the Finsler-Weyl connection of $(M,\mathcal{C},\alpha)$ and let $\omega^i_j = \sum \mathcal{K}^i_{jk}\, dx^k$ be the connections form of $(\mathcal{H},\mathcal{D})$. The curvature forms $\Omega^i_j = d\omega^i_j + \sum \omega^i_l \wedge \omega^l_j$ are given by

$$\Omega^i_j = \sum_{k<l} \mathcal{R}^i_{jkl}\, dx^k \wedge dx^l + \sum \mathcal{P}^i_{jkl}\, dx^k \wedge \theta^l,$$

where $\theta^l = dy^l + \sum \mathcal{N}^l_k \, dx^k$. The sets $\{dx^1, \cdots, dx^n\}$ and $\{\theta^1, \cdots, \theta^n\}$ form the dual basis for $\mathcal{H}^*$ and $\mathcal{V}^*$ respectively. The curvature tensors $\mathcal{R}^i_{jkl}$ and $\mathcal{P}^i_{jkl}$ are given by

$$\mathcal{R}^i_{jkl} = \sum \left(\frac{\partial}{\partial x^l}\right)^{\mathcal{H}} \mathcal{K}^i_{jk} - \sum \left(\frac{\partial}{\partial x^k}\right)^{\mathcal{H}} \mathcal{K}^i_{jl} + \sum \mathcal{K}^i_{mk}\mathcal{K}^m_{jl} - \sum \mathcal{K}^i_{ml}\mathcal{K}^m_{jk}, \tag{5.29}$$

and

$$\mathcal{P}^i_{jkl} = -\frac{\partial \mathcal{K}^i_{jk}}{\partial y^l}, \tag{5.30}$$

respectively. We say that $(\mathcal{H}, \mathcal{D})$ is *flat* if $\Omega^i_j = 0$.

Suppose that the Finsler-Weyl connection $(\mathcal{H}, \mathcal{D})$ is closed and flat. Then $\mathcal{P}^i_{jkl} = 0$ shows that the coefficients $\mathcal{K}^i_{jk}$ given by (5.12) are independent of the fiber coordinates $y^1, \cdots, y^n$. Therefore $(\mathcal{H}, \mathcal{D})$ is induced from a linear connection ∇ on TM, that is, $\mathcal{D}_{X^{\mathcal{H}}} Y^V = (\nabla_X Y)^V$. Then the Wagner connection $(\overline{\mathcal{H}}, \overline{\mathcal{D}})$ is also induced from a linear connection $\overline{\nabla}$ on TM. Theorem 5.5 shows that $\overline{\nabla}$ is the Lyra connection and ∇ the Weyl connection of (M, c, α) :

$$\mathcal{K}^i_{jk} = \left\{ {}^{\,i}_{j\,k} \right\} - \alpha_j \delta^i_k - \alpha_k \delta^i_j + \alpha^i g_{jk},$$

where $\alpha^i = \sum g^{ir} \alpha_r$ for the inverse (g^{ir}) of $g \in c$, and $\left\{ {}^{\,i}_{j\,k} \right\}$ are Christoffel symbols of g. Since ∇ is also flat and ∇ is symmetric, there exists a coordinate system $\{U, (x^i)\}_{1 \le i \le n}$ on M such that $\mathcal{K}^i_{jk}$ and $\mathcal{N}^i_k = \sum \mathcal{K}^i_{jk} y^k$ vanish on each U. Thus

$$\alpha_k = \frac{1}{n} \sum \left\{ {}^{\;m}_{m\,k} \right\} = \frac{1}{n} \frac{\partial}{\partial x^k} \log \sqrt{\det g} = \frac{1}{2} \frac{\partial}{\partial x^k} \log (\det g)^{1/n},$$

and the equation (5.15) is written as

$$\frac{\partial \left(e^{2\sigma_U} g_{ij} \right)}{\partial x^k} = 0, \qquad \sigma_U = -\log (\det g)^{1/n}. \tag{5.31}$$

Thus $e^{\sigma_U} L$ is independent of $x \in M$.

Theorem 5.6. *The conformal class $\mathcal{C}$ admits a conformally flat Finsler metric if and only if its Finsler-Weyl connection $(\mathcal{H}, \mathcal{D})$ is closed and flat.*

5.5 Conformally flat Randers metrics

From the discussion in the previous section, if a Finsler metric L is conformally flat, then the averaged Riemannian metric g is also conformally flat. Conversely in this section, we shall show that there exists a conformally flat Finsler metric if M is a hyperbolic space.

A vector field E on a Riemannian manifold (M, g) is said to be *semi-parallel* if

$$\nabla^g_X E = \rho\,(X + \varepsilon\beta(X)E) \tag{5.32}$$

for a constant ρ and $\varepsilon = \pm 1$, where β is the dual of E with respect g, that is, $\beta(X) = g(X, E)$. The integrability condition $\nabla^g_X \nabla^g_Y E - \nabla^g_Y \nabla^g_X E - \nabla^g_{[X,Y]} E = R^g(X, Y)$ for the existence of E satisfying (5.32) is given by

$$R^g(X, Y)E = -\varepsilon\rho^2\,[g(X, E)Y - g(Y, E)X]\,.$$

Hence the sectional curvature $K(X \wedge E)$ of the 2-plane $X \wedge E$ is given by

$$K(X \wedge E) = \frac{g\,(R^g(X, E)E, X)}{\|X\|^2\,\|X\|^2 - g(X, E)^2} = \varepsilon\rho^2.$$

Thus, if (M, g) is of constant curvature $K(X \wedge Y) = \varepsilon\rho^2$, the integrability condition is satisfied thus there exists a local semi-parallel vector field around every point of M.

Since ∇^g is compatible with g, this assumption (5.32) is equivalent to

$$\left(\nabla^g_X \beta\right)(Y) = \rho\,[g(X, Y) + \varepsilon\beta(X)\beta(Y)]\,.$$

Using this, we know that β is closed. Indeed,

$$\begin{aligned}
(d\beta)(X, Y) &= X(\beta(Y)) - Y(\beta(X)) - \beta([X, Y]) \\
&= X(\beta(Y)) - Y(\beta(X)) - \beta(\nabla^g_X Y - \nabla^g_Y X) \\
&= X(\beta(Y)) - \beta(\nabla^g_X Y) - \left[Y(\beta(X)) - \beta(\nabla^g_Y X)\right] \\
&= (\nabla^g_X \beta)(Y) - (\nabla^g_Y \beta)(X) \\
&= 0.
\end{aligned}$$

Lemma 5.1. *Let E be a semi-parallel vector field on a Riemannian manifold (M, g). Then its dual β with respect to g is closed.*

The condition (5.32) implies $\nabla^g_X \|E\|^2 = 2\rho\beta(X)(1 + \varepsilon\,\|E\|^2)$. Thus if E has constant length, E must be a unit vector field and $\varepsilon = -1$. Thus we replace the assumption (5.32) by

$$\nabla^g_X E = \rho\,[X - \beta(X)E]\,. \tag{5.33}$$

Then we define a linear connection $\overline{\nabla}$ by

$$\overline{\nabla}_X Y = \nabla^g_X Y + \rho[g(X,Y)E - \beta(Y)X]. \tag{5.34}$$

Then, by direct computation, we can show that $\overline{\nabla}$ is compatible with g, i.e., $\overline{\nabla} g = 0$ and $\overline{\nabla}_X Y - \overline{\nabla}_Y X - [X,Y] = \rho[\beta(X)Y - \beta(Y)X]$.

Lemma 5.2. *Let E be a vector unit vector field satisfying* (5.34) *on a Riemannian manifold (M,g), and let β be the dual of E with respect to g. The linear connection $\overline{\nabla}$ defined by* (5.34) *is a Lyra connection in (M,g).*

Furthermore

$$\begin{aligned}\overline{\nabla}_X E &= \nabla^g_X E + \rho[g(X,E)E - \beta(E)X]\\ &= \nabla^g_X E + \rho[\beta(X)E - X]\\ &= \rho[X - \beta(X)E] + \rho[\beta(X)E - X]\\ &= 0.\end{aligned}$$

Since $\overline{\nabla}$ is compatible with g, this means that the one-form β is also parallel with respect to $\overline{\nabla}$, i.e., $\overline{\nabla}_X\beta = 0$.

Proposition 5.3. *Let E be a semi-parallel vector field satisfying* (5.33)*, and let β be the dual of E with respect to g. Then the Randers metric*

$$L(X) = \sqrt{g(X,X)} + k\beta(X) \quad (0 < k < 1) \tag{5.35}$$

is preserved by the parallel translation with respect to $\overline{\nabla}$, that is, (M,L) is a Wagner space.

By direct calculations, the curvature $\overline{R}$ of the Lyra connection $\overline{\nabla}$ is computed as

$$\overline{R}(X,Y)Z = R^g(X,Y)Z + \rho^2\left[g(Y,Z)X - g(X,Z)Y\right]. \tag{5.36}$$

Lemma 5.3. *If (M,g) is of negative curvature $K = -\rho^2$, then $\overline{\nabla}$ is flat, i.e., $\overline{R} = 0$.*

Furthermore the connection ∇ defined by

$$\nabla_X Y := \overline{\nabla}_X Y - \rho\beta(X)Y = \nabla^g_X Y + \rho[g(X,Y)E - \beta(X)Y - \beta(Y)X] \tag{5.37}$$

is the Weyl connection with the Lee form $w_g = \rho\beta$, and since β is closed, the curvature R of ∇ coincides with $\overline{R}$. Especially, if (M,g) is of negative curvature $K = -\rho^2$, then the Weyl connection ∇ is also flat. Hence the Randers metric L given by (5.35) is conformally flat.

Example 5.1. ([Ai3]) Let $\mathbb{H} = \{(x^1, \cdots, x^n) \in \mathbb{R}^n | x^n > 0\}$ be the upper half plane with the Poincaré metric

$$g_P = \frac{1}{(x^n)^2} \sum dx^i \otimes dx^i.$$

The Christoffel symbols $\left\{ {i \atop jk} \right\}$ are given by

$$\left\{ {i \atop jk} \right\} = -\frac{1}{x^n} \left(\delta_{jn}\delta^i_k + \delta_{kn}\delta^i_j - \delta_{jk}\delta^i_n \right),$$

and $(\mathbb{H}, g_P)$ is negative constant curvature $K = -1$. The vector field

$$E = x^n \frac{\partial}{\partial x^n}$$

is a unit semi-parallel vector filed on $(\mathbb{H}, g_P)$. For the dual $\beta = \frac{1}{x^n} dx^n = d\log x^n$ of E, we define

$$L(X) = \sqrt{g_P(X, X)} + k\beta(X) = \frac{1}{x^n}\sqrt{\sum (X^i)^2} + k\frac{X^n}{x^n} \quad (0 < k < 1). \tag{5.38}$$

This Randers metric on $\mathbb{H}$ is conformally flat.

Bibliography

[Ai1] T. Aikou, Locally conformal Berwald spaces and Weyl structures, Publ.Math. Debrecen. **49**(1996), 113-120.

[Ai2] ______, Some remarks on Berwald manifolds and Landsberg manifold, Acta Math. Acad. Paedagog. NYháZi. **26**(2010), 139-148.

[Ai3] ______, Semi-parallel vector fields and conformally flat Randers metrics, Publ. Math. Debrecen. **78**(2011), 191-207.

[Ai4] ______, Averaged Riemannian metrics and connections with application to locally conformal Berwald manifolds, Publ. Math. Debrecen. **80**(2012), 179-198.

[Ai-Ic] T. Aikou and Y. Ichijyō, On Finsler-Weyl structures and conformal flatness, Rep. Fac. Sci. Kagoshima Univ. (Math., Phys. & Chem.), **23**(1990), 101-109.

[Ai-Ko] T. Aikou and L. Kozma, Global aspects of Finsler geometry, *in Handbook of Global Analysis* (edited by D. Krupka and D. Saunders), pp.1-39 (2008), Elsevier.

[Ba-Ch] D. Bao and S.S. Chern, A note on the Gauss-Bonnet theorem for Finsler space, Ann. Math. **143**(1996), 233-252.

[Ba-Ch-Sh] D. Bao, S.S. Chern and Z.Shen, *An Introduction to Riemann-Finsler Geometry*, GTM 200, Springer, 2000.

[Ba-Bo] P. Baum and R. Bott, Singularities of holomorphic foliations, J. Differential Geometry. **7**(1972), 279-342.

[Br] F. Brickell, A theorem on homogeneous functions, J. London Math. Soc. **42**(1967), 325-329.

[Ch-Ch-La] S. S. Chern, W. H. Chen and K. S. Lam, *Lectures On Differential Geometry*, World Scientific, 1999.

[De] A. Deicke, Über die Finsler-Räume mit $A_i = 0$, Arch. Math. **4**(1953), 45-51.

[Fo] G. B. Folland, Weyl manifolds, J. Differential Geometry. **4**(1970), 147-153.

[Ha] M. Hashiguchi, On conformal transformations of Finsler metrics, J. Math. Kyoto Univ. **16**(1976), 25-50.

[Ha-Ic] M. Hashiguchi and Y. Ichijyō, On conformal transformation in Wagner spaces, Rep. Fac. Sci. Kagoshima Univ. (Math., Phys. & Chem.), No **10**(1977), 19-25.

[Ha-Ai] Haripamyu and T. Aikou, Rizza structures in negative vector bundles, preprint, 2014.

[Ha-Sa-Sh] I. Hasegawa, V. S. Sabau, and H. Shimada, Randers spaces of constant flag curvature induced by almost contact metric structures, Hokkaido Math. J., **33**("004), 215-232.

[Ic] Y. Ichijyō, Finsler manifolds modeled on a Minkowski space, J. Math. Kyoto Univ. **16**(1976), 639-652.

[Ic-Ha] Y. Ichijyō and M. Hashiguchi, On the condition that a Randers space be conformally flat, Rep. Fac. Sci. Kagoshima Univ. (Math., Phys. & Chem.), No **22**(1989), 7-14.

[Je-Ai] Jenizon and Tadashi Aikou, Finsler-Weyl connections and Wagner connections of conformal Finsler manifolds, to appear in TENSOR, N.S. Vol. **76**, No.2 (2015).

[Ki] S. Kikuchi, On the condition that a space with (α, β)-metric be locally Minkowski, Tensor (N.S.), **33**(1979), 242-246.

[Ko] L. Kozma, On Finsler-Weyl manifolds and connections, Suppl. Rend. Palermo, Ser.II, No.**43**(1996), 173-179.

[La] S. Lang, *Fundamentals of Differential Geometry*, GTM, 191, Springer, 1998.

[Ma1] M. Matsumoto, *Foundations of Finsler Geometry and Special Finsler Spaces*, Kaiseisha-Press, Japan, 1986.

[Ma2] ———, On Finsler spaces with Randers' metric and special forms of important tensors, J. Math. Kyoto Univ., **14-3**(1974), 477-498.

[Ma-Ra-Tr-Ze] V. S. Matveev, H.-B. Radmacher, M. Troyanov and A. Zeghib, Finsler conformal Lichnerowicz-Obata conjecture, Ann. Inst. Fourier, Grenoble **59,** 3(2009), 937-949.

[Mo] X. Mo, *An Introduction to Finsler Geometry*, Series in Mathematics-Vol. 1, Peking University, 2006.

[Na-Ai] T. Nagano and T. Aikou, Dual connections in Finsler geometry, Acta Math. Acad. Paedagogicae Nyiregyhaziensis, No.**24** (2008), 103-114 .

[Ok] T. Okada, Minkowskian product of Finsler spaces and Berwald connection, J. Math. Kyoto Univ., **22**(1982), 323-332.

[Se-Va] D. K. Sen and J. R. Vanstone, On Weyl and Lyra manifolds, J. Math. Phsy., **13**(1972), 990-993.

[Sz-Lo-Ke] J. Szilasi, R. I. Lovas and D. Cs. Kertész, Several ways to a Berwald manifolds and some steps beyond, Extracta Mathematicae, Vol **26**(2011), 89–130.

[Sz1] Z. I. Szabó, Positive definite Berwald spaces, Tensor N. S. **35**(1981), 25-39.

[Sz2] ______, All Landsberg metrics are Landsberg , Ann. Glob. Anal. Geom. **34**(2008), 381-386.

[Sz3] ______, Correction to "All Landsberg metrics are Landsberg" , Ann. Glob. Anal. Geom. **35**(2009), 227-230.

[To-Et] R. G. Torrome and F. Etayo On a rigidity condition for Berwald spaces, RACSAM 104 (1), 2010, 69-80.

[Vi] Cs. Vincze, A new proof of Szabó's theorem on the Riemannian metrizability of Berwald manifolds, Acta Math. Acad. Paedagog. NYháZi. **21**(2005), 199–204.

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