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HIGHER RIESZ TRANSFORMS AND DERIVATIVES OF THE RIESZ KERNELS

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Using the Riesz transforms, derivatives of the Riesz kernels, homogeneous harmonic polynomials and derivatives of the Newton kernel, we introduce four kinds of the higher Riesz transforms. Further we study relations among them. As one of the preparations we investigate homogeneous polynomials which appear in numerators of derivatives of the Riesz kernels.

Keywords: Higher Riesz transforms, Riesz kernels, homogeneous polynomials.

Mathematics Subject Classification: 42B20, 31B99

1. INTRODUCTION AND PRELIMINARIES

Let $R^n (n \geq 2)$ be the n-dimensional Euclidean space. The points of R^n are ordered n-tuples $x = (x_1, \cdots, x_n)$, where each x_j is a real number. The term *multi-index* refers to an ordered n-tuple $\alpha = (\alpha_1, \cdots, \alpha_n)$ of nonnegative integers α_j. The multi-index e_j denotes the ordered n-tuple that has 1 in the jth spot and 0 everywhere else $(j = 1, \cdots, n)$. The following abbrebiated notations will be used: $\alpha_1 + \cdots + \alpha_n = |\alpha|, \alpha_1! \cdots \alpha_n! = \alpha!$ and $x_1^{\alpha_1} \cdots x_n^{\alpha_n} = x^\alpha$. For a nonnegative integer k, we denote $M_k = \{\alpha : |\alpha| = k\}$. We use the notations D_j and ∂_j for the pointwise differentiation with respect to x_j and the differentiation in the sense of distributions with respect to x_j, respectively. Moreover, for a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$ we set

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}, \quad \partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$$

and

$$\Delta = D_1^2 + \cdots + D_n^2, \tilde{\Delta} = \partial_1^2 + \cdots + \partial_n^2.$$

We introduce some function spaces. For a domain Ω the space $C^\infty(\Omega)$ denotes the space of all infinitely differentiable functions on Ω. The space $\mathcal{S}(R^n)$ is defined to be the class of all C^∞-functions φ on R^n such that

$$\sup_{x\in R^n} |x^\alpha D^\beta \varphi(x)| < \infty$$

for all multi-indices α and β. $\mathcal{S}(R^n)$ contains the space $\mathcal{D}(R^n)$ of all C^∞-functions with compact support. We let the space $\mathcal{S}(R^n)$ be equipped with its usual topology in distribution theory. The collection $\mathcal{S}'(R^n)$ of all continuous linear functionals on $\mathcal{S}(R^n)$ is called the space of tempered distributions. The pairing between distributions and test functions is denoted $< \cdot, \cdot >$. The Lebesgue spaces $L^1(R^n)$ and $L^2(R^n)$ are defined by

$$L^1(R^n) = \{f : ||f||_1 = \int_{R^n} |f(x)|dx < \infty\},$$

$$L^2(R^2) = \{f : ||f||_2 = (\int_{R^n} |f(x)|^2 dx)^{1/2} < \infty\}.$$

For a positive number r we set

$$Q^r(R^n) = \{f \in C^\infty(R^n) : (1+|x|)^r |D^\alpha f(x)| \text{ is bounded for each } \alpha\}$$

and

$$Q(R^n) = \cup_{r>0} Q^r(R^n).$$

The Fourier transform $\mathcal{F}_1 f$ in the L_1-sense of $f \in L^1(R^n)$ is defined by

$$\mathcal{F}_1 f(x) = \int e^{-ix\cdot y} f(y)dy$$

where $x \cdot y = x_1y_1 + \cdots + x_ny_n$. For $f \in \mathcal{D}(R^n)$

$$\mathcal{F}_1(D^\alpha f)(x) = (ix)^\alpha \mathcal{F}_1 f(x). \tag{1.1}$$

For $f \in L^2(R^n)$, we denote by $\mathcal{F}_2 f$ the Fourier transform of f in the L^2-sense. $\mathcal{F}_{S'}$ represents the Fourier transform in the sense of tempered distributions.

We denote by $2N$ the set of nonnegative even numbers. For a positive integer m, the Riesz kernel $\kappa_m(x)$ of order m is given by

$$\kappa_m(x) = \frac{1}{\gamma_{m,n}} \begin{cases} |x|^{m-n}, & m-n \notin 2N \\ (\delta_{m,n} - \log|x|)|x|^{m-n}, & m-n \in 2N \end{cases}$$

with

$$\gamma_{m,n} = \begin{cases} \pi^{n/2}2^n\Gamma(m/2)/\Gamma((n-m)/2), & m-n \notin 2N \\ (-1)^{(m-n)/2}2^{m-1}\pi^{n/2}\Gamma(m/2)((m-n)/2)!, & m-n \in 2N \end{cases}$$

and

$$\delta_{m,n} = \frac{\Gamma'(m/2)}{2\Gamma(m)} + \frac{1}{2}(1 + \frac{1}{2} + \cdots + \frac{1}{(m-n)/2} + C) - \log\pi$$

where C is Euler's constant. We note (see [Sc:§10 in Chap. VII]) that

$$\Delta^\ell \kappa_{2\ell}(x) = 0 \quad \text{for } x \neq 0, \tag{1.2}$$

$$\tilde{\Delta}^\ell \kappa_{2\ell} = (-1)^\ell \delta \tag{1.3}$$

where Δ^ℓ (resp. $\tilde{\Delta}^\ell$) is ℓ times iteration of Δ (resp. $\tilde{\Delta}$) and δ is the Dirac distribution. A function u is said to be polyharmonic of degree ℓ on a domain Ω if $\Delta^\ell u(x) = 0$ on Ω. So the Riesz kernel $\kappa_{2\ell}$ is polyharmonic of degree ℓ on $R^n - \{0\}$. Further the Fourier transform of κ_m is given by

$$\mathcal{F}_{S'}\kappa_m(x) = \text{Pf.}|x|^{-m} \tag{1.4}$$

where Pf. represents the pseudo function (see [Sc: §7 in Chap. VII]).

A function $k(x)$ on R^n is called a smooth Caldéron-Zygmund kernel if $k(x)$ satisfies the following three conditions:

(1.5) $k(x) \in C^\infty(R^n - \{0\})$,

(1.6) $k(x)$ is homogeneous of degree $-n$,

(1.7) $\int_{S_1} k(x)dS_1(x) = 0$

where S_1 is the unit sphere $\{|x| = 1\}$ and dS_1 is the surface element of S_1. For a smooth Caldéron-Zygmund kernel $k(x)$ we consider singular integral

$$Kf(x) = \lim_{\epsilon \to 0} \int_{|x-y| \geq \epsilon} k(x-y)f(y)dy.$$

We use the symbol C for a generic positive constant whose value may be different at each occurrence. By the L^2-theory of singular integrals [Sad: §2 in Chap. 6] we have

LEMMA 1.1. *For $f \in L^2(R^n)$,*

(i) *$Kf(x)$ exists for almost every $x \in R^n$,*

(ii) $||Kf||_2 \leq C||f||_2$,
(iii) $\mathcal{F}_2(Kf)(x) = \sigma(x)\mathcal{F}_2 f(x)$
where $\sigma(x)$ *is homogeneous of degree 0 and* $\int_{S_1} \sigma(x) dS_1(x) = 0$.

Moreover by [Ku] we have

LEMMA 1.2. *If* $f \in Q(R^n)$, *then* $Kf(x)$ *exists for every* $x \in R^n$ *and* $Kf \in Q(R^n)$.

It is clear that the functions $\frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \frac{x_j}{|x|^{n+1}}$ $(j = 1, \cdots, n)$ are smooth Calderón -Zygmund kernels. The singular integrals for the kernels $\frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \frac{x_j}{|x|^{n+1}}$ $(j = 1, \cdots, n)$ are called the Riesz transforms and denoted by R_j. Namely

$$R_j f(x) = \lim_{\epsilon \to 0} \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \int_{|x-y| \geq \epsilon} \frac{x_j - y_j}{|x-y|^{n+1}} f(y) dy.$$

The Fourier transform of $R_j f$ $(f \in L^2(R^n))$ is given by

$$\mathcal{F}_2(R_j f)(x) = \frac{-ix_j}{|x|} \mathcal{F}_2 f(x) \tag{1.8}$$

([Sad: §2 in Chap. 6]).

In this article we are concerned with the higher Riesz transforms. We introduce four kinds of the higher Riesz transforms. First, for a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$ we define R^α as follows:

$$R^\alpha = R_1^{\alpha_1} \cdots R_n^{\alpha_n}$$

(S.G.Samko [Sam:§4]). Secondly, we note that the kernels $\frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \frac{x_j}{|x|^{n+1}}$ are partial derivatives of the Riesz kernel $\kappa_1(x)$. Namely

$$\frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \frac{x_j}{|x|^{n+1}} = -D_j \kappa_1(x), \qquad x \neq 0.$$

We consider partial derivatives of order m of the Riesz kernel $\kappa_m(x)$. For a multi-index α, the partial derivative $D^\alpha \kappa_m(x)$ has the following form (Lemma 2.1): for $x \neq 0$

$$D^\alpha \kappa_m(x) = \begin{cases} \frac{P_{m,\alpha}(x)}{|x|^{n-m+2|\alpha|}}, & m - n \notin 2N \text{ or} \\ & m - n \in 2N, |\alpha| \geq m - n + 1 \\ \frac{-D^\alpha |x|^{m-n}}{\gamma_{m,n}} \log |x| + \frac{P_{m,\alpha}(x)}{|x|^{n-m+2|\alpha|}}, & m - n \in 2N, |\alpha| \leq m - n \end{cases}$$

where $P_{m,\alpha}(x)$ is a homogeneous polynomial of degree $|\alpha|$. Since $D^\alpha\kappa_m(x)$ is a smooth Caldéron-Zygmund kernel for $|\alpha| = m$ (Section 3, See also [Mi]), we can consider singular integral

$$S_m^\alpha f(x) = \lim_{\epsilon\to 0}\int_{|x-y|\geq\epsilon} D^\alpha\kappa_m(x-y)f(y)dy, \qquad |\alpha| = m.$$

Thirdly, we note that $\frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}}x_j$ is a homogeneous harmonic polynomial of degree 1. For a homogeneous harmonic polynomial $P(x)$ of degree m, it is clear that $\frac{P(x)}{|x|^{n+m}}$ is a smooth Caldéron-Zygmund kernel. Hence we can consider singular integral

$$T_m^P f(x) = \lim_{\epsilon\to 0}\int_{|x-y|\geq\epsilon}\frac{P(x-y)}{|x-y|^{n+m}}f(y)dy$$

(E.M.Stein [St: §3 in Chap. III]).

Finally, we note that $P_{2,\alpha}(x)$ is a homogeneous harmonic polynomial of degree $|\alpha|$ for any α (Theorem 3.7). Hence $\frac{P_{2,\alpha}(x)}{|x|^{n+|\alpha|}}$ is a smooth Caldéron-Zygmund kernel. So for any α we can consider singular integral

$$N^\alpha f(x) = \lim_{\epsilon\to 0}\int_{|x-y|\geq\epsilon}\frac{P_{2,\alpha}(x-y)}{|x-y|^{n+|\alpha|}}f(y)dy.$$

In section 2 we give relations between pointwise derivatives and ditributional derivarives of the Riesz kernels. In section 3 we study linear independence of $\{P_{m,\alpha} : \alpha \in M_k\}$ and polyharmonicity of $P_{2\ell,\alpha}$. In section 4 we state relations among R^α, S_m^α, T_m^P and N^α.

2. POINTWISE AND DITRIBUTIONAL DERIVATIVES OF THE RIESZ KERNELS

About pointwise partial derivatives of the Riesz kernels we note the following lemma, which is proved by induction and Leibniz's formula.

Lemma 2.1. *For $x \neq 0$, we have*

$$D^\alpha\kappa_m(x) = \begin{cases} \frac{P_{m,\alpha}(x)}{|x|^{n-m+2|\alpha|}}, & m-n \notin 2N \text{ or} \\ & m-n\in 2N, |\alpha| \geq m-n+1 \\ \frac{-D^\alpha|x|^{m-n}}{\gamma_{m,n}}\log|x| + \frac{P_{m,\alpha}(x)}{|x|^{n-m+2|\alpha|}}, & m-n\in 2N, |\alpha|\leq m-n \end{cases}$$

where $P_{m,\alpha}(x)$ is a homogeneous polynomial of degree $|\alpha|$.

The following lemma follows from Gauss's divergence theorem.

LEMMA 2.2. *Let Ω be a bounded domain with C^∞-boundary $\partial\Omega$. Let $\mathbf{n}(x) = (\mathbf{n}_1(x), \cdots, \mathbf{n}_n(x))$ denote the outer unit normal to the boundary at the point x of $\partial\Omega$. We assume that g and h have continuous partial derivatives on a neighborhood of the closure of Ω. Then*

$$\int_\Omega g(x)D_jh(x)dx = \int_{\partial\Omega} g(x)h(x)\mathbf{n}_j(x)dS(x) - \int_\Omega D_jg(x)h(x)dx$$

where dS represents the surface element of $\partial\Omega$.

LEMMA 2.3. *Let $\lambda \geq 1-n, g \in C^\infty(R^n - \{0\})$ be a homogeneous function of degree λ and $\varphi \in \mathcal{D}(R^n)$.*
(i) *If $\lambda > 1-n$, then*

$$\int g(x)D_j\varphi(x)dx = -\int D_jg(x)\varphi(x)dx, \tag{2.1}$$

$$\int g(x)(\log|x|)D_j\varphi(x)dx = -\int D_j(g(x)\log|x|)\varphi(x)dx. \tag{2.2}$$

(ii) *If $\lambda = 1-n$, then $\lim_{\epsilon\to 0}\int_{|x|\geq\epsilon} D_jg(x)\varphi(x)dx$ exists and*

$$\int g(x)D_j\varphi(x)dx = c_j\varphi(0) - \lim_{\epsilon\to 0}\int_{|x|\geq\epsilon} D_jg(x)\varphi(x)dx$$

where $c_j = -\int_{S_1} g(x)x_jdS_1(x)$.

Proof. (i) we give only a proof of (2.2) since the proof of (2.1) is similar. We set $S_\epsilon = \{x : |x| = \epsilon\}$ and dS_ϵ represents the surface element of S_ϵ. Since φ has compact support, by Lemma 2.2 we have

$$\begin{aligned} I_\epsilon &= \int_{|x|\geq\epsilon} g(x)(\log|x|)D_j\varphi(x)dx \\ &= -\int_{S_\epsilon} g(x)(\log|x|)\varphi(x)\mathbf{n}_j(x)dS_\epsilon(x) - \int_{|x|\geq\epsilon} D_j(g(x)\log|x|)\varphi(x)dx \end{aligned}$$

where $\mathbf{n}(x)$ is the outer unit normal at $x \in S_\epsilon$. Since $g(x)$ is homogeneous of degree λ, by the change of variables $x = \epsilon z$ we get

$$
\begin{aligned}
|\int_{S_\epsilon} g(x)(\log|x|)\varphi(x)\mathbf{n}_j(x)dS_\epsilon| &\le C\int_{S_\epsilon}|g(x)||\log|x||dS_\epsilon(x)\\
&= C\int_{S_1}|g(\epsilon z)||\log\epsilon|z||\epsilon^{n-1}dS_1(z)\\
&= C\epsilon^{\lambda+n-1}|\log\epsilon|\int_{S_1}|g(x)|dS_1(z)\to 0(\epsilon\to 0)
\end{aligned}
$$

because of $\lambda+n-1>0$. Since $g(x)(\log|x|)\varphi(x), D_j(g(x)\log|x|)\varphi(x)\in L^1(R^n)$, we obtain

$$\int g(x)(\log|x|)D_j\varphi(x)dx = \lim_{\epsilon\to 0} I_\epsilon = -\int D_j(g(x)\log|x|)\varphi(x)dx.$$

(ii) Let $\lambda = 1-n$. By Lemma 2.2 we have

$$
\begin{aligned}
J_\epsilon &= \int_{|x|\ge\epsilon} g(x)D_j\varphi(x)dx\\
&= -\int_{S_\epsilon} g(x)\varphi(x)\mathbf{n}_j(x)dS_\epsilon(x) - \int_{|x|\ge\epsilon} D_jg(x)\varphi(x)dx\\
&= -\int_{S_\epsilon} g(x)(\varphi(x)-\varphi(0))\mathbf{n}_j(x)dS_\epsilon(x) - \int_{S_\epsilon} g(x)\varphi(0)\mathbf{n}_j(x)dS_\epsilon(x) - \int_{|x|\ge\epsilon} D_jg(x)\varphi(x)dx\\
&= J_{1,\epsilon}+J_{2,\epsilon}+J_{3,\epsilon}
\end{aligned}
$$

Since $|\varphi(x)-\varphi(0)|\le C|x|$, the homogeneity of degree $1-n$ of g implies

$$J_{1,\epsilon}\to 0 \quad (\epsilon\to 0). \tag{2.3}$$

Moreover, since $\mathbf{n}_j(x) = x_j/|x|$ for $x\in S_\epsilon$, by homogeneity of degree $1-n$ of g we see that

$$J_{2,\epsilon} = -\varphi(0)\int_{S_1} g(x)x_jdS_1(x). \tag{2.4}$$

Since $g(x)D_j\varphi(x)$ is integrable, $\lim_{\epsilon\to 0} J_\epsilon$ exists, and hence $\lim_{\epsilon\to 0} J_{3,\epsilon}$ exists by (2.3) and (2.4). So we obtain

$$\int g(x)D_j\varphi(x)dx = -\varphi(0)\int_{S_1} g(x)x_jdS_1(x) - \lim_{\epsilon\to 0}\int_{|x|\ge\epsilon} D_jg(x)\varphi(x)dx.$$

This proves the lemma.

LEMMA 2.4. *Let k,m be positive integers with $k\ge m$ and $\varphi\in\mathcal{D}(R^n)$. We assume that for multi-indices β and γ with $|\beta|+|\gamma| = k, \gamma_j\ge 1$ and $|\gamma|\le k-m$,*

$$\lim_{\epsilon\to 0}\int_{|x|\ge\epsilon} D^\beta\kappa_m(x)D^\gamma(\varphi(x) - \sum_{|\delta|\le k-m-1}\frac{D^\delta\varphi(0)}{\delta!}x^\delta)dx$$

exists. Then

$$\lim_{\epsilon\to 0}\int_{|x|\geq\epsilon} D^{\beta+e_j}\kappa_m(x)D^{\gamma-e_j}(\varphi(x)-\sum_{|\delta|\leq k-m-1}\frac{D^\delta\varphi(0)}{\delta!}x^\delta)dx$$

exists, and

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_{|x|\geq\epsilon} D^{\beta}\kappa_m(x)D^{\gamma}(\varphi(x)-\sum_{|\delta|\leq k-m-1}\frac{D^\delta\varphi(0)}{\delta!}x^\delta)dx\\
&=\sum_{|\delta|=k-m,\delta\geq\gamma-e_j}\frac{-D^\delta\varphi(0)}{(\delta-(\gamma-e_j))!}\int_{S_1}D^\beta\kappa_m(x)x^{\delta-\gamma+2e_j}dS_1(x)\\
&\quad-\lim_{\epsilon\to 0}\int_{|x|\geq\epsilon} D^{\beta+e_j}\kappa_m(x)D^{\gamma-e_j}(\varphi(x)-\sum_{|\delta|\leq k-m-1}\frac{D^\delta\varphi(0)}{\delta!}x^\delta)dx.
\end{aligned}$$

Proof. First we note that the conditions $|\beta|+|\gamma|=k$ and $|\gamma|\leq k-m$ imply $|\beta|\geq m$. Hence by Lemma 2.1 $D^\beta\kappa_m(x)$ is homogeneous of degree $m-|\beta|-n$. Moreover

$$I_\epsilon=\int_{|x|\geq\epsilon} D^{\beta}\kappa_m(x)D^{\gamma}(\varphi(x)-\sum_{|\delta|\leq k-m-1}\frac{D^\delta\varphi(0)}{\delta!}x^\delta)dx$$

exists by the condition $|\beta|+|\gamma|=k$. So by Lemma 2.2 we have

$$\begin{aligned}
I_\epsilon &= \lim_{M\to\infty}\int_{\epsilon\leq|x|\leq M} D^{\beta}\kappa_m(x)D^{\gamma}(\varphi(x)-\sum_{|\delta|\leq k-m-1}\frac{D^\delta\varphi(0)}{\delta!}x^\delta)dx\\
&= \lim_{M\to\infty}\{\int_{S_M} D^{\beta}\kappa_m(x)D^{\gamma-e_j}(\varphi(x)-\sum_{|\delta|\leq k-m-1}\frac{D^\delta\varphi(0)}{\delta!}x^\delta)\mathbf{n}_j(x)dS_M(x)\\
&\quad-\int_{S_\epsilon} D^{\beta}\kappa_m(x)D^{\gamma-e_j}(\varphi(x)-\sum_{|\delta|\leq k-m-1}\frac{D^\delta\varphi(0)}{\delta!}x^\delta)\mathbf{n}_j(x)dS_\epsilon(x)\\
&\quad-\int_{\epsilon\leq|x|\leq M} D^{\beta+e_j}\kappa_m(x)D^{\gamma-e_j}(\varphi(x)-\sum_{|\delta|\leq k-m-1}\frac{D^\delta\varphi(0)}{\delta!}x^\delta)dx\}.
\end{aligned}$$

The condition $|\beta|+|\gamma|=k$ implies

$$\lim_{M\to\infty}\int_{S_M} D^{\beta}\kappa_m(x)D^{\gamma-e_j}(\varphi(x)-\sum_{|\delta|\leq k-m-1}\frac{D^\delta\varphi(0)}{\delta!}x^\delta)\mathbf{n}_j(x)dS_M(x)=0.$$

Hence we have

$$I_\epsilon = -\int_{S_\epsilon} D^{\beta}\kappa_m(x)D^{\gamma-e_j}(\varphi(x)-\sum_{|\delta|\leq k-m}\frac{D^\delta\varphi(0)}{\delta!}x^\delta)\mathbf{n}_j(x)dS_\epsilon(x)$$

$$
\begin{aligned}
& -\int_{S_\epsilon} D^\beta \kappa_m(x) D^{\gamma-e_j}\Big(\sum_{|\delta|=k-m} \frac{D^\delta\varphi(0)}{\delta!} x^\delta\Big)\mathbf{n}_j(x)dS_\epsilon(x) \\
& -\int_{|x|\geq\epsilon} D^{\beta+e_j}\kappa_m(x) D^{\gamma-e_j}\Big(\varphi(x) - \sum_{|\delta|\leq k-m-1} \frac{D^\delta\varphi(0)}{\delta!} x^\delta\Big)dx \\
= \; & I_{1,\epsilon} + I_{2,\epsilon} + I_{3,\epsilon}
\end{aligned}
$$

Taylor's formula and homogeneity of degree $m-|\beta|-n$ of $D^\beta\kappa_m(x)$ give

$$
\begin{aligned}
|I_{1,\epsilon}| &\leq C\int_{S_\epsilon} |D^\beta\kappa_m(x)||x|^{k-m-|\gamma|+2} dS_\epsilon(x) \\
&= C\epsilon \int_{S_1} |D^\beta\kappa_m(x)| dS_1(x) \to 0 \quad (\epsilon\to 0).
\end{aligned}
$$

Moreover, by the change of variables $x=\epsilon z$ and homogeneity of degree $m-|\beta|-n$ of $D^\beta\kappa_m(x)$ we have

$$
\begin{aligned}
I_{2,\epsilon} &= -\sum_{|\delta|=k-m,\delta\geq\gamma-e_j} \frac{D^\delta\varphi(0)}{(\delta-(\gamma-e_j))!}\int_{S_\epsilon} D^\beta\kappa_m(x) x^{\delta-(\gamma-e_j)}\mathbf{n}_j(x)dS_\epsilon(x) \\
&= -\sum_{|\delta|=k-m,\delta\geq\gamma-e_j} \frac{D^\delta\varphi(0)}{(\delta-(\gamma-e_j))!}\int_{S_1} D^\beta\kappa_m(z) z^{\delta-\gamma+2e_j} dS_1(z).
\end{aligned}
$$

This completes the proof of the lemma.

Lemma 2.5. *Let k and m be positive integers.*
(i) *If $k<m$ and $\sum_{\alpha\in M_k} c_\alpha D^\alpha\kappa_m(x) = f(x)$ for $x\neq 0$, then $\sum_{\alpha\in M_k} c_\alpha \partial^\alpha\kappa_m = f$.*
(ii) *If $k\geq m$ and $\sum_{\alpha\in M_k} c_\alpha D^\alpha\kappa_m(x) = 0$ for $x\neq 0$, then $\sum_{\alpha\in M_k} c_\alpha\partial^\alpha\kappa_m$ is a lenear combination of $\partial^\beta\delta$ $(\beta\in M_{k-m})$.*

Proof. Let $\varphi\in\mathcal{D}(R^n)$. We have

$$
\begin{aligned}
I &= <\sum_{\alpha\in M_k} c_\alpha\partial^\alpha\kappa_m, \varphi> = (-1)^k\sum_{\alpha\in M_k} c_\alpha < \kappa_m, D^\alpha\varphi> \\
&= (-1)^k\sum_{\alpha\in M_k} c_\alpha\int \kappa_m(x)D^\alpha\varphi(x)dx = (-1)^k\sum_{k\in M_k} c_\alpha I_\alpha.
\end{aligned}
$$

First, let $k<m$ and $\sum_{\alpha\in M_k} c_\alpha D^\alpha\kappa_m(x) = f(x)$ for $x\neq 0$. Since $m-(k-1)-n > 1-n$, by applying Lemma 2.3 (i) repeatedly we obtain

$$
I = \sum_{\alpha\in M_k} c_\alpha\int D^\alpha\kappa_m(x)\varphi(x)dx = \int\Big(\sum_{\alpha\in M_k} c_\alpha D^\alpha\kappa_m(x)\Big)\varphi(x)dx.
$$

Therefore, the assumption gives $I = < f, \varphi >$. This proves (i). Next, let $k \geq m$ and $\sum_{\alpha \in M_k} c_\alpha D^\alpha \kappa_m(x) = 0$ for $x \neq 0$. We write α as follows: $\alpha = e_{j_1} + \cdots + e_{j_k}$. Since $m - s - n > 1 - n$ for $s < m - 1$, by applying Lemma 2.3 (i) repeatedly we have

$$\begin{aligned} I_\alpha &= \int \kappa_m(x) D^{e_{j_1} + \cdots + e_{j_k}} \varphi(x) dx \\ &= (-1)^{m-1} \int D^{e_{j_1} + \cdots + e_{j_{m-1}}} \kappa_m(x) D^{e_{j_m} + \cdots + e_{j_k}} \varphi(x) dx. \end{aligned}$$

Since $D^{e_{j_1} + \cdots + e_{j_{m-1}}} \kappa_m(x)$ is homogeneous of degree $1 - n$, by applying Lemma 2.3 (ii) we see that

$$\lim_{\epsilon \to 0} \int_{|x| \geq \epsilon} D^{e_{j_1} + \cdots + e_{j_m}} \kappa_m(x) D^{e_{j_{m+1}} + \cdots + e_{j_k}} \varphi(x) dx$$

exists, and

$$\begin{aligned} I_\alpha &= (-1)^{m-1} \{ C_{j_m} D^{e_{j_{m+1}} + \cdots + e_{j_k}} \varphi(0) \\ &\quad - \lim_{\epsilon \to 0} \int_{|x| \geq \epsilon} D^{e_{j_1} + \cdots + e_{j_m}} \kappa_m(x) D^{e_{j_{m+1}} + \cdots + e_{j_k}} \varphi(x) dx \}. \end{aligned}$$

Further, by applying Lemma 2.4 repeatedly we have

$$\begin{aligned} &\lim_{\epsilon \to 0} \int_{|x| \geq \epsilon} D^{e_{j_1} + \cdots + e_{j_m}} \kappa_m(x) D^{e_{j_{m+1}} + \cdots + e_{j_k}} \varphi(x) dx \\ &= \lim_{\epsilon \to 0} \int_{|x| \geq \epsilon} D^{e_{j_1} + \cdots + e_{j_m}} \kappa_m(x) D^{e_{j_{m+1}} + \cdots + e_{j_k}} (\varphi(x) - \sum_{|\eta| \leq k-m-1} \frac{D^\eta \varphi(0)}{\eta!} x^\eta) dx \\ &= \sum_{\beta \in M_{k-m}} d_{\alpha,\beta} D^\beta \varphi(0) + (-1)^{k-m} \lim_{\epsilon \to 0} \int_{|x| \geq \epsilon} D^\alpha \kappa_m(x) (\varphi(x) - \sum_{|\eta| \leq k-m-1} \frac{D^\eta \varphi(0)}{\eta!} x^\eta) dx \end{aligned}$$

with suitable constants $d_{\alpha,\beta}$. Consequently, by the assumption for suitable constants d_β we obtain

$$\begin{aligned} I &= (-1)^k \sum_{\alpha \in M_k} c_\alpha I_\alpha \\ &= (-1)^k \sum_{\beta \in M_{k-m}} d_\beta D^\beta \varphi(0) \\ &\quad + \lim_{\epsilon \to 0} \int_{|x| \geq \epsilon} (\sum_{\alpha \in M_k} c_\alpha D^\alpha \kappa_m(x)) (\varphi(x) - \sum_{|\eta| \leq k-m-1} \frac{D^\eta \varphi(0)}{\eta!} x^\eta) dx \\ &= (-1)^k \sum_{\beta \in M_{k-m}} d_\beta D^\beta \varphi(0) = < \sum_{\beta \in M_{k-m}} (-1)^m d_\beta \partial^\beta \delta, \varphi > . \end{aligned}$$

This completes the proof of (ii).

We use the following properties of pseudo functions in the next section.

LEMMA 2.6. *Let ℓ be a real number and $P(x)$ be a homogeneous function. Then*

$$P(x)\mathrm{Pf.}\frac{1}{|x|^{\ell}} = \mathrm{Pf.}\frac{P(x)}{|x|^{\ell}}.$$

LEMMA 2.7. $\partial_j \mathrm{Pf.}|x|^{-n} = \mathrm{Pf.}D_j|x|^{-n} + \omega\partial_j\delta$ *where* $\omega = -\int_{S_1} y_1^2 dS_1(y) = -\frac{2\pi^{n/2}}{n\Gamma(n/2)}$.

3. HOMOGENEOUS POLYNOMIALS IN DERIVATIVES OF THE RIESZ KERNELS

We let $\mathcal{P}_k (k \geq 1)$ be the set of all homogeneous polynomials of degree k. The dimension of $\mathcal{P}_k$ is

$$\binom{n+k-1}{k} = \frac{(n+k-1)!}{(n-1)!k!}.$$

We note that $P_{m,\alpha} \in \mathcal{P}_k$ for $\alpha \in M_k$. We denote by $V_{m,k}$ the set of all finite linear combinations of elements belonging to the set $\{P_{m,\alpha} : \alpha \in M_k\}$.

THEOREM 3.1. *Let k, m be positive integers and $k < m$. If $m - n \notin 2N$ or $m - n \in 2N, k \geq m - n - 1$, then the elements of the set $\{P_{m,\alpha} : \alpha \in M_k\}$ are linearly independent.*

Proof. Let $\sum_{\alpha\in M_k} c_\alpha P_{m,\alpha}(x) = 0$. First, let $m - n \notin 2N$ or $m - n \in 2N, k \geq m - n + 1$. By Lemma 2.1, for $x \neq 0$ we have

$$0 = \sum_{\alpha\in M_k} \frac{c_\alpha P_{m,\alpha}(x)}{|x|^{n-m+2k}} = \sum_{\alpha\in M_k} c_\alpha D^\alpha \kappa_m(x).$$

Lemma 2.5 (i) gives

$$\sum_{\alpha\in M_k} c_\alpha \partial^\alpha \kappa_m = 0.$$

By taking the Fourier transforms $\mathcal{F}_{S'}$ of the both sides we get

$$\sum_{\alpha\in M_k} c_\alpha (ix)^\alpha \mathrm{Pf.}|x|^{-m} = 0$$

and hence $\sum_{\alpha\in M_k} c_\alpha x^\alpha = 0$. This implies that $c_\alpha = 0$ for all $\alpha \in M_k$.

Next, let $m - n \in 2N$ and $k = m - n$. By Lemma 2.1, for $x \neq 0$ we have

$$\begin{aligned}\sum_{\alpha\in M_k} c_\alpha D^\alpha \kappa_m(x) &= \sum_{\alpha\in M_k} c_\alpha\left(\frac{-D^\alpha |x|^{m-n}}{\gamma_{m,n}} \log|x| + \frac{P_{m,\alpha}(x)}{|x|^{n-m+2k}}\right) \\ &= b\log|x|\end{aligned}$$

where b is a constant. Hence Lemma 2.5 (i) gives

$$\sum_{\alpha \in M_k} c_\alpha \partial^\alpha \kappa_m(x) = b \log |x|.$$

By taking the Fourier transforms $\mathcal{F}_{S'}$ we obtain

$$\begin{aligned} \sum_{\alpha \in M_k} c_\alpha (ix)^\alpha \mathrm{Pf.}|x|^{-m} &= b\mathcal{F}_{S'}(\log |x|) \\ &= b(c_1 \mathrm{Pf.}|x|^{-n} + c_2 \delta) \end{aligned}$$

where

$$c_1 = -2^{n-1}\Gamma(n/2)\pi^{n/2}, \quad c_2 = (2\pi)^n(-\frac{\mathcal{C}}{2} + \frac{\Gamma'(n/2)}{2\Gamma(n/2)} - \log \pi)$$

(see [Sc: §7 in Chap. VII]). Hence Lemma 2.6 gives

$$(\sum_{\alpha \in M_k} c_\alpha (ix)^\alpha - c_1 b |x|^{m-n}) \mathrm{Pf.}|x|^{-m} = c_2 b \delta.$$

Therefore for $\varphi \in \mathcal{D}$ with supp$\varphi \subset R^n - \{0\}$, we have

$$\begin{aligned} 0 &= < (\sum_{\alpha \in M_k} c_\alpha (ix)^\alpha - c_1 b |x|^{m-n}) \mathrm{Pf.}|x|^{-m}, \varphi > \\ &= \int \frac{\sum_{\alpha \in M_k} c_\alpha (ix)^\alpha - c_1 b |x|^{m-n}}{|x|^m} \varphi(x) dx. \end{aligned}$$

The arbitrariness of φ implies

$$\sum_{\alpha \in M_k} c_\alpha (ix)^\alpha - c_1 b |x|^{m-n} = 0 \quad \text{on} \quad R^n - \{0\},$$

and hence

$$\sum_{\alpha \in M_k} c_\alpha (ix)^\alpha - c_1 b |x|^{m-n} = 0 \quad \text{on} \quad R^n.$$

This gives

$$(\sum_{\alpha \in M_k} c_\alpha (ix)^\alpha - c_1 b |x|^{m-n}) \mathrm{Pf.}|x|^{-m} = 0$$

and so

$$c_2 b \delta = 0.$$

Hence we have

$$b = 0,$$

and

$$\sum_{\alpha\in M_k} c_\alpha (ix)^\alpha \mathrm{Pf.}|x|^{-m} = 0.$$

This implies that $c_\alpha = 0$ for all $\alpha \in M_k$. Finally we let $m-n \in 2N$ and $k = m-n-1(\geq 1)$. By Lemma 2.1, for $x \neq 0$ we have

$$\begin{aligned}
\sum_{\alpha\in M_k} c_\alpha D^\alpha \kappa_m(x) &= \sum_{\alpha\in M_k} c_\alpha (\frac{-D^\alpha |x|^{m-n}}{\gamma_{m,n}} \log|x| + \frac{P_{m,\alpha}(x)}{|x|^{n-m+2k}}) \\
&= \sum_{j=1}^n d_j x_j \log|x|.
\end{aligned}$$

By taking the Fourier transforms $\mathcal{F}_{S'}$ of the both sides we obtain

$$\begin{aligned}
\sum_{\alpha\in M_k} c_\alpha (ix)^\alpha \mathrm{Pf.}|x|^{-m} &= \sum_{j=1}^n d_j \mathcal{F}_{S'}(x_j \log|x|) \\
&= i\sum_{j=1}^n d_j \partial_j \mathcal{F}_{S'}(\log|x|) \\
&= i\sum_{j=1}^n d_j \partial_j (c_1 \mathrm{Pf.}|x|^{-n} + c_2\delta) \\
&= i\sum_{j=1}^n d_j (c_1(\mathrm{Pf.}\frac{-nx_j}{|x|^{n+2}} + \omega\partial_j\delta) + c_2\partial_j\delta) \\
&= ic_1 \sum_{j=1}^n d_j \mathrm{Pf.}\frac{-nx_j}{|x|^{n+2}} + i\sum_{j=1}^n d_j(c_1\omega + c_2)\partial_j\delta
\end{aligned}$$

where we used Lemma 2.7. Hence by Lemma 2.6 we have

$$(\sum_{\alpha\in M_k} c_\alpha (ix)^\alpha - ic_1\sum_{j=1}^n d_j(-nx_j)|x|^{m-n-2})\mathrm{Pf.}|x|^{-m} = i\sum_{j=1}^n d_j(c_1\omega + c_2)\partial_j\delta.$$

Therefore, for $\varphi \in \mathcal{D}$ and $\mathrm{supp}\varphi \subset R^n - \{0\}$, we obtain

$$\begin{aligned}
0 &= < (\sum_{\alpha\in M_k} c_\alpha (ix)^\alpha - ic_1\sum_{j=1}^n d_j(-nx_j)|x|^{m-n-2})\mathrm{Pf.}|x|^{-m}, \varphi > \\
&= \int \frac{\sum_{\alpha\in M_k} c_\alpha (ix)^\alpha - ic_1\sum_{j=1}^n d_j(-nx_j)|x|^{m-n-2}}{|x|^m}\varphi(x)dx.
\end{aligned}$$

Since φ is arbitrary, we see that

$$\sum_{\alpha\in M_k} c_\alpha (ix)^\alpha - ic_1\sum_{j=1}^n d_j(-nx_j)|x|^{m-n-2} = 0 \quad \text{on} \quad R^n - \{0\}$$

and hence

$$\sum_{\alpha\in M_k} c_\alpha (ix)^\alpha - ic_1 \sum_{j=1}^{n} d_j(-nx_j)|x|^{m-n-2} = 0 \quad \text{on} \quad R^n.$$

This gives

$$(\sum_{\alpha\in M_k} c_\alpha (ix)^\alpha - ic_1 \sum_{j=1}^{n} d_j(-nx_j)|x|^{m-n-2})\text{Pf.}|x|^{-m} = 0$$

and so

$$\sum_{j=1}^{n} d_j(c_1\omega + c_2)\partial_j\delta = 0.$$

Since $c_1\omega + c_2 \neq 0$ ($c_1\omega + c_2$ is an increasing function of n) , we have $d_j = 0$ for $j = 1, 2, \cdots, n$. Therefore

$$\sum_{\alpha\in M_k} c_\alpha (ix)^\alpha \text{Pf.}|x|^{-m} = 0.$$

This implies that $c_\alpha = 0$ for all $\alpha \in M_k$. This proves the theorem.

For a multi-index β we set

$$M_k + \beta = \{\alpha + \beta : \alpha \in M_k\}.$$

Further, for a set $E \subset M_k$, $M_k \setminus E$ means

$$M_k \setminus E = \{\alpha \in M_k : \alpha \notin E\}.$$

THEOREM 3.2. *Let $k \geq m$.*
(i) *If m is an odd number, then the elements of the set $\{P_{m,\alpha} : \alpha \in M_k\}$ are linearly independent.*
(ii) *If m is an even number 2ℓ, then for each $\eta \in M_\ell$, the elements of the set $\{P_{2\ell,\alpha} : \alpha \in M_k \setminus (M_{k-2\ell} + 2\eta)\}$ are linearly independent.*

Proof. (i) Let $\sum_{\alpha\in M_k} c_\alpha P_{m,\alpha}(x) = 0$. Since

$$0 = \sum_{\alpha\in M_k} \frac{c_\alpha P_{m,\alpha}(x)}{|x|^{n-m+2k}} = \sum_{\alpha\in M_k} c_\alpha D^\alpha \kappa_m(x)$$

for $x \neq 0$ by Lemma 2.1 and $k \geq m$, Lemma 2.5 (ii) gives

$$\sum_{\alpha\in M_k} c_\alpha \partial^\alpha \kappa_m = \sum_{\beta\in M_{k-m}} d_\beta \partial^\beta \delta.$$

By taking the Fourier transforms $\mathcal{F}_{S'}$ of the both sides we get

$$\sum_{\alpha\in M_k} c_\alpha (ix)^\alpha \text{Pf.}|x|^{-m} = \sum_{\beta\in M_{k-m}} d_\beta (ix)^\beta.$$

Hence

$$\sum_{\alpha\in M_k} c_\alpha (ix)^\alpha = \sum_{\beta\in M_{k-m}} d_\beta (ix)^\beta |x|^m. \tag{3.1}$$

Since m is an odd number, the equality (3.1) implies that the both sides of (3.1) are zero. Hence $c_\alpha = 0$ for all $\alpha \in M_k$.

(ii) Let $\sum_{\alpha\in M_k\setminus(M_{k-2\ell}+2\eta)} c_\alpha P_{2\ell,\alpha}(x) = 0$ for $\eta \in M_\ell$. Then for $x \neq 0$

$$0 = \sum_{\alpha\in M_k\setminus(M_{k-2\ell}+2\eta)} \frac{c_\alpha P_{2\ell,\alpha}(x)}{|x|^{n-2\ell+2k}} = \sum_{\alpha\in M_k\setminus(M_{k-2\ell}+2\eta)} c_\alpha D^\alpha \kappa_{2\ell}(x)$$

by Lemma 2.1 and $k \geq 2\ell$. Since $k \geq 2\ell$, Lemma 2.5 (ii) gives

$$\sum_{\alpha\in M_k\setminus(M_{k-2\ell}+2\eta)} c_\alpha \partial^\alpha \kappa_{2\ell}(x) = \sum_{\beta\in M_{k-2\ell}} d_\beta \partial^\beta \delta.$$

By taking Fourier transforms $\mathcal{F}_{S'}$ of both sides we obtain

$$\sum_{\alpha\in M_k\setminus(M_{k-2\ell}+2\eta)} c_\alpha (ix)^\alpha \mathrm{Pf}.|x|^{-2\ell} = \sum_{\beta\in M_{k-2\ell}} d_\beta (ix)^\beta.$$

Hence

$$\begin{aligned}\sum_{\alpha\in M_k\setminus(M_{k-2\ell}+2\eta)} c_\alpha (ix)^\alpha &= \sum_{\beta\in M_{k-2\ell}} d_\beta (ix)^\beta (x_1^2+\cdots+x_n^2)^\ell \\ &= \sum_{\beta\in M_{k-2\ell}} (-1)^\ell d_\beta \frac{\ell!}{\eta!}(ix)^{\beta+2\eta} + \sum_{\beta\in M_{k-2\ell}} d_\beta (ix)^\beta \sum_{\gamma\in M_\ell,\gamma\neq\eta} \frac{\ell!}{\gamma!} x^{2\gamma}.\end{aligned}$$

Since the left side does not contain the term $x^{\beta+2\eta}(\beta \in M_{k-2\ell})$, we see that $d_\beta = 0$ for $\beta \in M_{k-2\ell}$, and hence the right side is zero. Consequently, $\sum_{\alpha\in M_k\setminus(M_{k-2\ell}+2\eta)} c_\alpha x^\alpha = 0$ and hence $c_\alpha = 0$ for all $\alpha \in M_k \setminus (M_{k-2\ell}+2\eta)$. This proves the theorem.

Let ℓ be a positive integer. By (1.2) the Riesz kernel $\kappa_{2\ell}(x)$ is polyharmonic of order ℓ on $R^n - \{0\}$. Hence for a multi-index α,

$$D^\alpha \kappa_{2\ell}(x) = \begin{cases} \frac{P_{2\ell,\alpha}(x)}{|x|^{n-2\ell+2|\alpha|}}, & 2\ell - n \notin 2N \text{ or} \\ & 2\ell - n \in 2N, |\alpha| \geq 2\ell - n + 1 \\ \frac{-D^\alpha |x|^{2\ell-n}}{\gamma_{2\ell,n}} \log|x| + \frac{P_{2\ell,\alpha}(x)}{|x|^{n-2\ell+2|\alpha|}}, & 2\ell - n \in 2N, |\alpha| \leq 2\ell - n \end{cases}$$

is polyharmonic of order ℓ on $R^n - \{0\}$. Further we show that $P_{2\ell,\alpha}(x)$ is polyharmonic of order ℓ on R^n. We need some lemmas. The following lemma follows from straightforward computation and Euler's formula for homogeneous functions.

LEMMA 3.3. *Let r, s be real numbers and $u(x) \in C^\infty(R^n - \{0\})$ be homogeneous of degree r. Then for $x \neq 0$*

$$\Delta(|x|^s u(x)) = |x|^s \Delta u(x) + s(s+2r+n-2)|x|^{s-2} u(x).$$

Let k, ℓ and m be positive integers with $k \leq \ell$. For $0 \leq j \leq k$, we set

$$\begin{aligned} C_j^{k,\ell,m} &= 2^j \binom{k}{j} (n-2\ell+2m)(n-2\ell+2m+2)\cdots(n-2\ell+2m+2(j-1)) \\ &\quad \times (k-\ell)(k-(\ell+1))\cdots(k-(\ell+(j-1))), \qquad j=1,\cdots,k \end{aligned}$$

and

$$C_0^{k,\ell,m} = 1.$$

The coefficients $C_j^{k,\ell,m}$ have the following properties which are verified straightforwardly.

LEMMA 3.4. *Let $k+1 \leq \ell$. Then*

(i) $C_j^{k,\ell,m} + C_{j-1}^{k,\ell,m} 2(n-2\ell+2m+2(j-1))(2k+1-\ell-(j-1)) = C_j^{k+1,\ell,m}$.

(ii) $$C_k^{k,\ell,m} 2(n-2\ell+2m+2k)(k+1-\ell) = C_{k+1}^{k+1,\ell,m}.$$

LEMMA 3.5. *Let k, ℓ, m be positive integers with $k \leq \ell$ and $u \in C^\infty(R^n - \{0\})$ be homogeneous of degree m. Then for $x \neq 0$*

$$\Delta^k\left(\frac{u(x)}{|x|^{n-2\ell+2m}}\right) = \sum_{j=0}^{k} C_j^{k,\ell,m} \frac{\Delta^{k-j} u(x)}{|x|^{n-2\ell+2m+2j}}. \tag{3.2}$$

Proof. For $k=1$, (3.2) follows from Lemma 3.3. We assume that (3.2) holds for $k(\leq \ell-1)$. By the assumption of induction, for $x \neq 0$ we have

$$\begin{aligned} \Delta^{k+1}\left(\frac{u(x)}{|x|^{n-2\ell+2m}}\right) &= \Delta\left(\Delta^k\left(\frac{u(x)}{|x|^{n-2\ell+2m}}\right)\right) \\ &= \Delta\left(\sum_{j=0}^{k} C_j^{k,\ell,m} \frac{\Delta^{k-j} u(x)}{|x|^{n-2\ell+2m+2j}}\right). \end{aligned}$$

Further, by Lemma 3.3 we see that for $x \neq 0$

$$\begin{aligned} &\Delta^{k+1}\left(\frac{u(x)}{|x|^{n-2\ell+2m}}\right) \\ &= \sum_{j=0}^{k} C_j^{k,\ell,m} \left\{ \frac{\Delta^{k+1-j} u(x)}{|x|^{n-2\ell+2m+2j}} + 2(n-2\ell+2m+2j)(2k+1-\ell-j) \frac{\Delta^{k-j} u(x)}{|x|^{n-2\ell+2m+2j+2}} \right\} \end{aligned}$$

$$
\begin{aligned}
= & \frac{\Delta^{k+1}u(x)}{|x|^{n-2\ell+2m}} \\
& + \sum_{j=1}^{k}\{C_j^{k,\ell,m} + C_{j-1}^{k,\ell,m}2(n-2\ell+2m+2(j-1))(2k+1-\ell-(j-1))\}\frac{\Delta^{k+1-j}u(x)}{|x|^{n-2\ell+2m+2j}} \\
& + C_k^{k,\ell,m}2(n-2\ell+2m+2k)(k+1-\ell)\frac{u(x)}{|x|^{n-2\ell+2m+2j+2}}.
\end{aligned}
$$

Therefore Lemma 3.4 gives

$$\Delta^{k+1}\left(\frac{u(x)}{|x|^{n-2\ell+2m}}\right) = \sum_{j=0}^{k+1} C_j^{k+1,\ell,m}\frac{\Delta^{k+1-j}u(x)}{|x|^{n-2\ell+2m+2j}}$$

for $x \neq 0$, and hence we obtain (3.2) for $k+1$.

COROLLARY 3.6. *Let ℓ, m be positive integers and $u \in C^\infty(R^n - \{0\})$ be homogeneous of degree m. Then for $x \neq 0$*

$$\Delta^{\ell}\left(\frac{u(x)}{|x|^{n-2\ell+2m}}\right) = \frac{\Delta^{\ell}u(x)}{|x|^{n-2\ell+2m}}.$$

Proof. This corollary follows from the fact that $C_j^{\ell,\ell,m} = 0$ for $j = 1, \cdots, \ell$.

THEOREM 3.7. *$P_{2\ell,\alpha}(x)$ is polyharmonic of degree ℓ on R^n.*

Proof. Since $P_{2\ell,\alpha}$ is a homogeneous polynomial of degree $|\alpha|$, the theorem is clear for $|\alpha| < 2\ell$. Let $|\alpha| \geq 2\ell$. By (1.2) and Lemma 2.1, for $x \neq 0$ we have

$$0 = D^\alpha \Delta^\ell \kappa_{2\ell}(x) = \Delta^\ell D^\alpha \kappa_{2\ell}(x) = \Delta^\ell\left(\frac{P_{2\ell,\alpha}(x)}{|x|^{n-2\ell+2|\alpha|}}\right).$$

Moreover, since $P_{2\ell,\alpha}$ is a homogeneous of degree $|\alpha|$, Corollary 3.6 gives

$$0 = \frac{\Delta^\ell P_{2\ell,\alpha}(x)}{|x|^{n-2\ell+2|\alpha|}}$$

for $x \neq 0$. This implies that $\Delta^\ell P_{2\ell,\alpha}(x) = 0$ for $x \neq 0$, and hence $P_{2\ell,\alpha}$ is polyharmonic of degree ℓ on R^n.

We denote by $\mathcal{A}_\ell$ the set of all polyharmonic functions of degree ℓ on R^n. By Theorem 3.7 we have $V_{2\ell,k} \subset \mathcal{P}_k \cap \mathcal{A}_\ell$. We show that $V_{2\ell,k} = \mathcal{P}_k \cap \mathcal{A}_\ell$, and give a basis of the vector space $V_{2\ell,k}$. The following lemma is due to (E.M.Stein and G.Weiss [SW: §2 in Chap. IV]). For a real number r we denote by $[r]$ the integral part of r.

LEMMA 3.8. *If $P \in \mathcal{P}_k$, then*

$$P(x) = P_0(x) + |x|^2 P_1(x) + \cdots + |x|^{2k_1} P_{k_1}(x)$$

where $k_1 = [k/2]$ and P_j is a homogeneous harmonic polynomial of degree $k-2j, j = 0, 1, \cdots, k_1$.

LEMMA 3.9. *Let j, s be positive integers with $j > s$ and u be a homogeneous harmonic function on R^n. Then*

$$\Delta^j(|x|^{2s}u(x)) = 0.$$

Proof. Let the degree of homogeneity of u be r. By Lemma 3.3 we have

$$\Delta^s(|x|^{2s}u(x)) = c(r, s)u(x)$$

where $c(r, s) = 2^s s!(2s+2r+n-2)(2(s-1)+2r+n-2)\cdots(2+2r+n-2)$. Hence for $j > s$ we obtain $\Delta^j(|x|^{2s}u(x)) = 0$.

LEMMA 3.10. *Let k, ℓ be positive integers, $k_1 = [k/2]$ and P be a homogeneous polynomial of degree k. Then P is polyharmonic of degree ℓ if and only if*

$$P(x) = P_0(x) + |x|^2 P_1(x) + \cdots + |x|^{2s} P_s(x)$$

where $s = \min(\ell - 1, k_1)$ and P_j is a homogeneous harmonic polynomial of degree $k - 2j, j = 0, 1, \cdots, s$.

Proof. If $k = 1$, then the lemma is obvious. Let $k \geq 2$. If $\ell - 1 \geq k_1$, then $2\ell > k$. Hence the lemma follows from Lemma 3.8 and the fact that a homogeneous polynomial of degree k is polyharmonic of degree ℓ. Let $\ell - 1 < k_1$. If $P(x) = P_0(x) + |x|^2 P_1(x) + \cdots + |x|^{2s} P_s(x)$, then $\Delta^\ell P(x) = 0$ by Lemma 3.9 since $s = \ell - 1$. Conversely, we assume that P is polyharmonic of degree ℓ. Since P is a homogeneous polynomial of degree k, by Lemma 3.8

$$P(x) = P_0(x) + |x|^2 P_1(x) + \cdots + |x|^{2k_1} P_{k_1}(x)$$

where P_j is a homogeneous harmonic polynomial of degree $k-2j, j=0,1,\cdots,k_1$. By the assumption, $\Delta^\ell P(x)=\Delta^{\ell+1}P(x)=\cdots=\Delta^{k_1}P(x)=0$. By Lemma 3.9, we see that $0=\Delta^{k_1}P(x)=c(k-2k_1,k_1)P_{k_1}(x)$. Since $c(k-2k_1,k_1)\neq 0$, This implies that $P_{k_1}=0$. By repeating the above procedure we obtain that $P_\ell=P_{\ell+1}=\cdots=P_{k_1}=0$. This proves the lemma.

Corollary 3.11. *Let P be a homogeneous polynomial of degree 2ℓ. Then P is polyharmonic of degree ℓ if and only if*

$$\int_{S_1} P(x)dS_1(x)=0.$$

Proof. By Lemma 3.8 we have

$$P(x)=P_0(x)+|x|^2P_1(x)+\cdots+|x|^{2(\ell-1)}P_{\ell-1}(x)+c_\ell|x|^{2\ell}$$

where P_j is a homogeneous harmonic polynomial of degree $2\ell-2j, j=0,1,\cdots,\ell-1$ and c_ℓ is a constant. By harmonicity of P_j and $P_j(0)=0, j=0,1,\cdots,\ell-1$, we see that

$$\int_{S_1} P(x)dS_1(x)=c_\ell\sigma_n.$$

where $\sigma_n=\int_{S_1}dS_1(x)$. Hence the corollary follows from Lemma 3.10.

Lemma 3.12. (E.M.Stein and G.Weiss [SW: §2 in Chap. IV])

$$\dim(\mathcal{P}_k\cap\mathcal{A}_1)=\begin{cases}\begin{pmatrix}n+k-1\\k\end{pmatrix}-\begin{pmatrix}n+k-3\\k-2\end{pmatrix}, & k\geq 2\\ n, & k=1.\end{cases}$$

Corollary 3.13. *Let k and ℓ be positive integers. Then*

$$\dim(\mathcal{P}_k\cap\mathcal{A}_\ell)=\begin{cases}\begin{pmatrix}n+k-1\\k\end{pmatrix}-\begin{pmatrix}n+k-2\ell-1\\k-2\ell\end{pmatrix}, & k\geq 2\ell\\ \begin{pmatrix}n+k-1\\k\end{pmatrix}, & k<2\ell.\end{cases}$$

Proof. This corollary follows from Lemmas 3.10 and 3.12.

Now we have

THEOREM 3.14. (I) *Let* $k < m$. *If* $m-n \notin 2N$ *or* $m-n \in 2N, k \geq m-n-1$, *then* $V_{m,k} = \mathcal{P}_k$ *and* $\{P_{m,\alpha} : \alpha \in M_k\}$ *is a basis of* $V_{m,k}$.
(II) *Let* $k \geq m$.
(i) *If* m *is an odd number, then* $V_{m,k} = \mathcal{P}_k$ *and* $\{P_{m,\alpha} : \alpha \in M_k\}$ *is a basis of* $V_{m,k}$.
(ii) *If* m *is an even number* 2ℓ, *then* $V_{2\ell,k} = \mathcal{P}_k \cap \mathcal{A}_\ell$ *and for each* $\eta \in M_\ell$, $\{P_{2\ell,\alpha} : \alpha \in M_k \setminus (M_{k-2\ell} + 2\eta)\}$ *is a basis of* $V_{2\ell,k}$.

Proof. (I) This follows from Theorem 3.1 and dim $V_{m,k} =$ dim $\mathcal{P}_k$.
(II)(i) This follows from Theorem 3.2(i) and dim $V_{m,k}$ =dim $\mathcal{P}_k$.
(II)(ii) $V_{2\ell,k} \subset \mathcal{P}_k \cap \mathcal{A}_\ell$ follows from Theorem 3.7. Hence Theorem 3.2(ii) and Corollary 3.13 implies that dim $V_{2\ell,k}$ =dim $(\mathcal{P}_k \cap \mathcal{A}_\ell)$. Therefore $V_{2\ell,k} = \mathcal{P}_k \cap \mathcal{A}_\ell$ and $\{P_{2\ell,\alpha} : \alpha \in M_k \setminus (M_{k-2\ell} + 2\eta)\}$ is a basis of $V_{2\ell,k}$.

4. HIGHER RIESZ TRANSFORMS

In this section we state relations among the four kinds of the higher Riesz transforms. As defined in section 1, for a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$ we set

$$R^\alpha = R_1^{\alpha_1} \cdots R_n^{\alpha_n}.$$

By (1.8), for $f \in L^2(R^n)$

$$\mathcal{F}_2(R^\alpha f)(x) = \frac{(-i)^{|\alpha|} x^\alpha}{|x|^{|\alpha|}} \mathcal{F}_2 f(x). \tag{4.1}$$

The Riesz potential U_m^f of order m of $f \in \mathcal{D}(R^n)$ is defined by

$$U_m^f(x) = \int \kappa_m(x-y) f(y) dy.$$

M.Ohtsuka [Oh] proved that $D_j U_1^f(x) = -R_j f(x)$. Moreover we have

LEMMA 4.1. *Let $f \in \mathcal{D}(R^n)$ and $|\alpha| = m$. Then*

$$R^\alpha f(x) = (-1)^m U_m^{D^\alpha f}(x) = (-1)^m D^\alpha U_m^f(x) \tag{4.2}$$

for all $x \in R^n$.

Proof. By (1.1), (1.4) and (4.1) we have

$$\mathcal{F}_{S'}(U_m^{D^\alpha f}) = \mathcal{F}_{S'}(D^\alpha U_m^f) = (-1)^m \mathcal{F}_{S'}(R^\alpha f) = \frac{(ix)^\alpha}{|x|^m}\mathcal{F}_1 f.$$

Since $R^\alpha f \in C^\infty(R^n)$ by Lemma 1.2 and $U_m^{D^\alpha f}, D^\alpha U_m^f \in C^\infty(R^n)$, (4.2) holds for all $x \in R^n$.

We note that the equality $R^\alpha f = (-1)^m D^\alpha U_m^f$ is shown in the sense of weak derivatives in [Sam1: Theorem 7] and [Sam3: Theorem 7.25].

In case m is an odd number, for $|\alpha| = m$ $D^\alpha \kappa_m(x) = \frac{P_{m,\alpha}(x)}{|x|^{n+m}}$ is obviously a smooth Calderón-Zygmund kernel. In case m is an even number 2ℓ, for $|\alpha| = 2\ell$ $D^\alpha \kappa_{2\ell}(x) = \frac{P_{2\ell,\alpha}}{|x|^{n+2\ell}}$ is also a smooth Calderón-Zygmund kernel by Theorem 3.7 and Corollary 3.11. We set

$$S_m^\alpha f(x) = \lim_{\epsilon\to 0}\int_{|x-y|\geq\epsilon} D^\alpha \kappa_m(x-y) f(y)dy, \quad |\alpha| = m.$$

In case $m = 2$, by Theorem 3.7 $P_{2,\alpha}$ is a homogeneous harmonic polynomial of degree $|\alpha|$ for any α. Hence $\frac{P_{2,\alpha}(x)}{|x|^{n+|\alpha|}}$ is a smooth Calderón-Zygmund kernel. So for any α we can consider singular integral

$$N^\alpha f(x) = \lim_{\epsilon\to 0}\int_{|x-y|\geq\epsilon} \frac{P_{2,\alpha}(x-y)}{|x-y|^{n+|\alpha|}} f(y)dy.$$

THEOREM 4.2. *Let $f \in L^2(R^n)$ and $|\alpha| = m$. Then*

$$R^\alpha f = (-1)^m S_m^\alpha f + c_{m,\alpha} f$$

where

$$c_{m,\alpha} = \frac{(-i)^m}{\sigma_n}\int_{S_1} x^\alpha dS_1(x).$$

Proof. It suffices to show the theorem for $f \in \mathcal{D}(R^n)$. We write $\alpha = e_{j_1} + \cdots + e_{j_m}$. We have

$$S_m^\alpha f(x) = \lim_{\epsilon\to 0}\int_{|y|\geq\epsilon} D_{j_m} D^{\alpha - e_{j_m}} \kappa_m(y) f(x-y)dy.$$

Since $D^{\alpha-e_{j_m}}\kappa_m(y)$ is homogeneous of degree $1-n$, Lemma 2.3 (ii) gives

$$S_m^\alpha f(x) = d_{m,\alpha}f(x) + \int D^{\alpha-e_{j_m}}\kappa_m(y)D_{j_m}f(x-y)dy.$$

where $d_{m,\alpha}$ is a suitable constant. Since $D^{\alpha-e_{j_m}-e_{j_{m-1}}}\kappa_m(y), D^{\alpha-e_{j_m}-e_{j_{m-1}}-e_{j_{m-2}}}\kappa_m(y)$, $\cdots, D^{e_{j_1}}\kappa_m(y)$ are homogeneous functions of degree more than $1-n$ or products of homogeneous functions of degree more than $1-n$ and $\log|x|$, by applying Lemma 2.3 (i) repeatedly we obtain

$$S_m^\alpha f(x) = d_{m,\alpha}f(x) + \int \kappa_m(y)D^\alpha f(x-y)dy = d_{m,\alpha}f(x) + U_m^{D^\alpha f}(x).$$

Hence Lemma 4.1 implies

$$R^\alpha f(x) = (-1)^m S_m^\alpha f(x) + c_{m,\alpha}f(x) \tag{4.3}$$

where $c_{m,\alpha} = (-1)^{m+1}d_{m,\alpha}$. By taking the Fourier transforms $\mathcal{F}_2$ of the both sides, and using (4.1) and Lemma 1.1 we obtain

$$\frac{(-i)^m x^\alpha}{|x|^m}\mathcal{F}_2 f(x) = \sigma_\alpha(x)\mathcal{F}_2 f(x) + c_{m,\alpha}\mathcal{F}_2 f(x)$$

where σ_α is homogeneous of degree 0 and $\int_{S_1}\sigma_\alpha(x)dS_1(x) = 0$. Therefore

$$\frac{(-i)^m x^\alpha}{|x|^m} = \sigma_\alpha(x) + c_{m,\alpha}.$$

This gives

$$c_{m,\alpha} = \frac{(-i)^m}{\sigma_n}\int_{S_1} x^\alpha dS_1(x).$$

This completes the proof of the theorem.

Y.Mizuta [Mi] and S.G.Samko [Sam2] give the value of the integral $\int_{S_1} x^\alpha dS_1(x)$. Namely

$$\int_{S_1} x^\alpha dS_1(x) = \frac{2\prod_{j=1}^n \frac{1+(-1)^{\alpha_j}}{2}\Gamma(\frac{\alpha_j+1}{2})}{\Gamma(\frac{n+|\alpha|}{2})}.$$

Hence

Lemma 4.3. *Let $\alpha = (\alpha_1, \cdots, \alpha_n)$ be a multi-index. Then $\int_{S_1} x^\alpha dS_1(x) = 0$ if and only if there exists i such that α_i is an odd number.*

By Lemmas 4.1, 4.3 and Therorem 4.2 we have

COROLLARY 4.4. *Let $f \in \mathcal{D}(R^n)$ and α be a multi-index. Then $D^\alpha U_m^f = S_m^\alpha f$ if and only if there exists i such that α_i is an odd number.*

Since $\frac{P(x)}{|x|^{n+k}}$ is a smooth Calderon-Zygmund kernel for a homogeneous harmonic polynomial $P(x)$ of degree k, we can consider singular integral

$$T_k^P f(x) = \lim_{\epsilon \to 0} \int_{|x-y|\geq \epsilon} \frac{P(x-y)}{|x-y|^{n+k}} f(y)dy.$$

THEOREM 4.5. *Let α be a multi-index with $|\alpha| = m$. Then*

$$S_m^\alpha = \sum_{j=0}^{\ell} T_{m-2j}^{P_j}$$

where $\ell = [(m-1)/2]$ and P_j is a homogeneous harmonic polynomial of degree $m-2j$, $(j = 0, 1, \cdots, \ell)$.

Proof. By Theorem 3.7 and Lemmas 3.8, 3.10, we have

$$P_{m,\alpha}(x) = P_0(x) + |x|^2 P_1(x) + \cdots + |x|^{2\ell} P_\ell(x)$$

where $\ell = [(m-1)/2]$ and P_j is a homogeneous harmonic polynomial of degree $m-2j$, $(j = 0, 1, \cdots, \ell)$. Hence

$$\frac{P_{m,\alpha}(x)}{|x|^{n+m}} = \sum_{j=0}^{\ell} \frac{P_j(x)}{|x|^{n+m-2j}}.$$

This proves the theorem.

Finally we use Theorem 3.14 for $m = 2$. Then we have

THEOREM 4.6. *Let P be a homogeneous harmonic polynomial of degree k. Then*

$$T_k^P = \begin{cases} \sum_{\alpha \in M_k} c_\alpha N^\alpha, & k = 1 \\ \sum_{\alpha \in M_k \setminus (M_{k-2} + 2e_1)} c_\alpha N^\alpha, & k \geq 2. \end{cases}$$

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