INFINITESIMAL MIXED TORELLI PROBLEM FOR ALGEBRAIC SURFACES WITH ORDINARY SINGULARITIES, I

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INFINITESIMAL MIXED TORELLI PROBLEM FOR ALGEBRAIC SURFACES WITH ORDINARY SINGULARITIES, I*

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Dedicated to the memory of N. Sasakura

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ABSTRACT. In this paper, we formulate the *infinitesimal mixed Torelli problem* for an algebraic surface S with ordinary singularities. We use 2-cubic hyper-resolution $a_\bullet : X_\bullet \longrightarrow S$ ($\bullet \in \square_2$) of S in the sense of F. Guillén, V. Navarro Aznar *et al.* not only to describe the mixed Hodge structure on the cohomology of S, but also to describe the *infinitesimal locally trivial deformation space* $H^1(S, \Theta_S)$ of S, where $\Theta_S := \mathcal{H}om_{\mathcal{O}_S}(\Omega^1_S, \mathcal{O}_S)$. For an analytic family $\pi : \mathfrak{S} \longrightarrow (M, o)$ of *locally trivial deformations* of S, parametrized by a pointed complex space (M, o), we define the *Kodaira-Spencer map* $\sigma_o : T_oM \longrightarrow H^1(S, \Theta_S)$. We show that if each fiber of the family $\pi : \mathfrak{S} \longrightarrow (M, o)$ is projective, then the variation of mixed Hodge structures, arising from this family, can be described by taking 2-cubic hyper-resolution of its each fiber simultaneously. We give a formula which describes the relation between the Kodaira-Spencer map $\sigma_o : T_oM \longrightarrow H^1(S, \Theta_S)$ and the Jacobian map $d\Phi_o$ of the so-called *period map* $\Phi : M \longrightarrow \mathcal{M}_{mix}(H^1(S)_\mathbb{Z}) \times \mathcal{M}_{mix}(H^2(S)_\mathbb{Z})$ at $o \in M$, where $\mathcal{M}_{mix}(H^\ell(S)_\mathbb{Z})$, $\ell = 1, 2$, denotes the *modular variety of mixed Hodge structures* on $H^\ell(S)_\mathbb{Z}$ $:= H^\ell(S, \mathbb{Z})$ modulo torsion (Theorem 3.17).

Key words: Surface, Ordinary singularity, Cubic hyper-resolution, Mixed Hodge structure, Infinitesimal mixed Torelli

Contents

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Introduction

In this paper, we shall study the infinitesimal mixed Torelli problem for algebraic surfaces with ordinary singularities defined over the complex number field. This kind of surfaces is attractive because every smooth algebraic surface defined over the complex number field is obtained as the normalization of such a surface in the three dimensional complex projective space $\mathbb{P}^3(\mathbb{C})$. Indeed, it is well known that every smooth algebraic surface embedded in a complex projective space of sufficiently higher dimension can be projected onto such a surface in $\mathbb{P}^3(\mathbb{C})$ via generic projection.

Thanks to Deligne's result ([3]) there exist *mixed Hodge structures* on the cohomology groups of a *singular* complex projective variety. Hence we may consider *infinitesimal mixed Torelli problem* for singular complex projective varieties like infinitesimal Torelli problem for non-singular complex projective ones. However, for this purpose, we need to consider an *equisingular* family of singular complex projective varieties in some sense so that from such a family there arises naturally a *variation of mixed Hodge structure*. In this paper, we shall consider a *locally trivial* complex analytic family of (complex projective) *algebraic surfaces with ordinary singularities*. Describing the mixed Hodge structure on the cohomology groups of an algebraic surface with ordinary singularities by use of its *cubic hyper-resolution* in the sense of F. Guillén, V. Navarro Aznar *et al* ([13]), we shall formulate the infinitesimal mixed Torelli problem for such a surface, and give sufficient conditions for the problem to be affirmatively solved. The arrangement of this paper is as follows:

In §1, for the reader's convenience, we shall review the definition of cubic hyper-resolution of an algebraic variety due to F. Guillén, V. Navarro Aznar *et al.*, and construct a natural 2-cubic hyper-resolution $a_\bullet : X_\bullet \longrightarrow S$ for an algebraic surface S with ordinary singularities. Further, we shall explain how the mixed Hodge structures on the cohomology groups of S can be described in terms of this 2-cubic hyper-resolution.

In §2 we shall describe the variation of mixed Hodge structures arising from a *locally trivial complex analytic family* $\pi : \mathfrak{S} \longrightarrow M$ of algebraic surfaces with ordinary singularities, parametrized by a complex manifold M, using the simultaneous 2-cubic hyper-resolution $\mathfrak{X}_\bullet \xrightarrow{b_\bullet} \mathfrak{S} \xrightarrow{\pi} M$ of each fiber $S_t := \pi^{-1}(t)$, $t \in M$, of the family $\pi : \mathfrak{S} \longrightarrow M$. The simultaneous 2-cubic hyper-resolution $\mathfrak{X}_\bullet \xrightarrow{b_\bullet} \mathfrak{S} \xrightarrow{\pi} M$ will be generalized to the notion of a *cubic hyper-equisingular family of complex projective varieties*. In §2 we shall work in this more general frame work.

In §3 we shall define the *Kodaira-Spencer* map

$$\sigma_o : T_oM \longrightarrow H^1(S, \Theta_S)$$

for a complex analytic family $(\mathfrak{S}, \pi, M, o, \phi)$ of locally trivial deformations of a compact complex surface $S \simeq S_o := \pi^{-1}(o)$, $o \in M$, with ordinary singularities, parametrized by a pointed complex space (M, o); and when M is non-singular and each fiber of the family $\mathfrak{S}$ is projective, we give a formula which describes the relation between the Kodaira-Spencer map $\sigma_o : T_oM \longrightarrow H^1(S, \Theta_S)$ and the Jacobian map

$$d\Phi_o : T_oM \longrightarrow \oplus_{\ell=1}^{2}\{\oplus_{p=1}^{\ell}\mathrm{Hom}_{\mathbb{C}}(\mathbb{H}^{\ell-p}(\Omega_{X_\bullet}^{p}[1]), \mathbb{H}^{\ell-p+1}(\Omega_{X_\bullet}^{p-1}[1]))\}$$

of the so-called *period map*

$$\Phi : M \longrightarrow \mathcal{M}_{mix}(H^1(S)_{\mathbb{Z}})) \times \mathcal{M}_{mix}(H^2(S)_{\mathbb{Z}}))$$

at $o \in M$, which comes from the variation of mixed Hodge structures on $R^{\ell}_{\mathcal{O}}(\pi) := R^{\ell}\pi_*\mathbb{Z} \otimes \mathcal{O}_M$, $\ell = 1, 2$, where $\mathcal{M}_{mix}(H^{\ell}(S)_{\mathbb{Z}}))$, $\ell = 1, 2$, denotes the *modular variety of mixed Hodge structures* on $H^{\ell}(S)_{\mathbb{Z}} := H^{\ell}(S, \mathbb{Z})$ modulo torsion (Theorem 3.16). For this purpose, we need not only to describe the variation of mixed Hodge structures arising from the family $(\mathfrak{S}, \pi, M, o, \phi)$ by use of the simultaneous 2-cubic hyper-resolution $\mathfrak{X}_\bullet \xrightarrow{b_\bullet} \mathfrak{S} \xrightarrow{\pi} M$ of the family $\pi : \mathfrak{S} \longrightarrow M$, but also to describe $H^1(S, \Theta_S)$ by use of the 2-cubic hyper-resolution $a_\bullet : X_\bullet \longrightarrow S$. We shall define the sheaf $\Theta(a_\bullet)$ of germs of holomorphic tangent vector fields to the 2-cibic hyper-resolution $a_\bullet : X_\bullet \longrightarrow S$, which is a coherent sheaf on S with the property that

$$H^1(S, \Theta_S) \simeq H^1(S, \Theta(a_\bullet)),$$

and define a homomorphism

$$\rho_o : T_oM \longrightarrow H^1(S, \Theta(a_\bullet)),$$

which might be considered as the *Kodaira-Spencer map* of the simultaneous 2-cubic hyper-resolution $\mathfrak{X}_\bullet \xrightarrow{b_\bullet} \mathfrak{S} \xrightarrow{\pi} M$ of the family $\pi : \mathfrak{S} \longrightarrow M$. Then we can show that there exists a homomorphism

$$\tau_o : H^1(S, \Theta(a_\bullet)) \longrightarrow \oplus_{\ell=1}^{2}\{\oplus_{p=1}^{\ell}\mathrm{Hom}_{\mathbb{C}}(\mathbb{H}^{\ell-p}(\Omega_{X_\bullet}^{p}[1]), \mathbb{H}^{\ell-p+1}(\Omega_{X_\bullet}^{p-1}[1]))\},$$

and that the composite $\tau_o \circ \rho_o$ coincides with the Jacobian map $d\Phi_o$ of the period map Φ at $o \in M$ up to sign.

In §4 we shall describe the hyper-cohomology $\mathbb{H}^\bullet(\Omega_{X_\bullet}^\bullet)$ in terms of the cohomology groups of the non-singular objects X_α $(\alpha \in \square_2)$, of the 2-cubic hyper-resolution $X_\bullet \longrightarrow S$. Further, under some condition, we shall express the cohomology $H^1(S, \Theta_S)$ in terms of the cohomology groups of the non-singular objects X_α $(\alpha \in \square_2)$ with the coefficients of the sheaf of germs of *descendable* holomorphic vector fields on X_α. Note that the condition above is satisfied if the genus of every irreducible components of D_X^*, the normal model of $D_X := f^{-1}(D_S)$, where D_S is the double curve of S and $f : X \longrightarrow S$ the normalization map of

S, is greater than one. After these works, we shall reformulate the infinitesimal mixed Torelli problem for an algebraic surface S with ordinary singularities in the form where we can handle the problem more easily cohomologically, and give sufficient conditions for the problem to be affimatively solved.

In §5, with an aid of the results in §4, we shall give a few examples of algebraic surfaces with ordinary singularities for which the infinitesimal mixed Torelli problem is affirmatively solved.

We would like to end the introduction with two comments:

(1) In this paper, we have not taken into consideration "polarization" of the family. Hence the family we consider in this paper is not a "variation of gradedly polarized mixed Hodge structure", but just a "variation of mixed Hodge structure".

(2) Throughout this paper our method is basically "complex analytic", and we always consider algebraic manifolds and algebraic varieties defined over the complex number field as comlex manifolds and comlex spaces.

Owing to its length, this paper is divided into two parts: Part I includes sections 1-3, and Part II sections 4-5.

§1 Algebraic surfaces with ordinary singularities, their cubic hyper-resolutions, and mixed Hodge structures on their cohomology

1.1 Definition. A 2-dimensional compact complex surface S is said to be *with ordinary singularities* if it is locally isomorphic to one of the following germs of hypersurfaces at the origin of the complex 3-space $\mathbb{C}^3$ at every point of S :

$$\begin{cases} (i)\ z=0\ \text{(simple point)}, & (ii)\ yz=0\ \text{(ordinary double point)}, \\ (iii)\ xyz=0\ \text{(ordinary triple point)}, & (iv)\ xy^2-z^2=0\ \text{(cuspidal point)}, \end{cases}$$

where (x,y,z) is the coordinate on $\mathbb{C}^3$. Further, if S is projective, we call it an *algebraic surface with ordinary singularities*.

We denote by D_S the singular locus of S, and call it the *double curve* of S. D_S is a singular curve with triple points. We denote by Σt_S the triple point locus of S, and by $\Sigma \mathfrak{c}_S$ the cuspidal point locus of S.

In order to describe the mixed Hodge structure on $H^\ell(S,\mathbb{C})$ $(0\leq\ell\leq4)$, we construct a *cubic hyper-resolution* of S in the sense of V. Navarro Aznar, F. Guillén et al. by taking normalizations successively. First, we refer to some notions from [13]. For a non-negative integer n, let $\square_n^+$ the *augmented n-cubic category*, i.e., the category whose objects $\mathrm{Ob}(\square_n^+)$ and the set of homomorphisms $\mathrm{Hom}_{\square_n^+}(\alpha,\beta)$ $(\alpha=(\alpha_0,\alpha_1,\cdots,\alpha_n),\beta=(\beta_0,\beta_1,\cdots,\beta_n)$ $\in \mathrm{Ob}(\square_n^+))$ are given as follows:

$$\mathrm{Ob}(\square_n^+) := \{\alpha=(\alpha_0,\alpha_1,\cdots,\alpha_n)\in\mathbb{Z}^{n+1} \mid 0\leq\alpha_i\leq1 \text{ for } 0\leq i\leq n\},$$

$$\mathrm{Hom}_{\square_n^+}(\alpha,\beta) := \begin{cases} \alpha\to\beta \text{ (an arrow from } \alpha \text{ to } \beta) & \text{if } \alpha_i\leq\beta_i \text{ for } 0\leq i\leq n \\ \emptyset & \text{otherwise.} \end{cases}$$

For $n = -1$ we define $\square_{-1}^{+}$ to be the punctual category $\{*\}$, i.e., the category consisting of a single point. For $n \geq 0$ the *n-cubic category*, denoted by $\square_n$, is defined to be the full subcategory of $\square_n^+$ with $\mathrm{Ob}(\square_n) = \mathrm{Ob}(\square_n^+) - \{(0, \cdots, 0)\}$. Notice that $\mathrm{Ob}(\square_n^+)$ can be considered as a finite ordered set whose order is defined by $\alpha \leq \beta \Longleftrightarrow \alpha \to \beta$ for $\alpha, \beta \in \mathrm{Ob}(\square_n^+)$.

1.2 Definition. A $\square_n^+$-object (resp. $\square_n$-object) of a category $\mathcal{C}$ is a contravariant functor $X_\bullet^+$ (resp. $X_\bullet$) from $\square_n^+$ (resp. $\square_n$) to $\mathcal{C}$. It is also called an *augmented n-cubic object of $\mathcal{C}$* (resp. an *n-cubic object of $\mathcal{C}$*).

1.3 Definition. Let $X_\bullet, Y_\bullet$ be $\square_n^+$-objects (resp. $\square_n$-objects) of a category $\mathcal{C}$. We define a morphism $\Phi_\bullet : X_\bullet \to Y_\bullet$ to be a natural transformation from the functor $X_\bullet$ to the one $Y_\bullet$ over the identity functor $\mathrm{id} : \square_n^+ \to \square_n^+$ (resp. $\mathrm{id} : \square_n \to \square_n$).

1.4 Definition Let $X_\bullet$ be an n-cubic object of $\mathcal{C}$ ($n \geq 0$), X a (-1)-object of $\mathcal{C}$. An *augmentation of $X_\bullet$ to X* is a natural transformation from the functor $X_\bullet$ to the one $X \times \square_n$ over the natural functor $\square_n \to \square_{-1}^+ = \{*\}$.

We may think of an n-cubic object of $\mathcal{C}$ with an augmentation to X as an augmented n-cubic object of $\mathcal{C}$. Conversely, an augmented n-cubic object $X_\bullet^+ : (\square_n^+)^\circ \to \mathcal{C}$ of $\mathcal{C}$ can be identified with an n-cubic object $X_\bullet := X_{\bullet|\square_n}^+ : (\square_n)^\circ \to \mathcal{C}$ of $\mathcal{C}$ with an augmentation to $X_{(0,\cdots,0)}^+$, where $\circ$ denotes the dual category. In what follows we shall interchangeably use an augmented n-cubic object of $\mathcal{C}$ and an n-cubic object of $\mathcal{C}$ with an augmentation.

1.5 Definition A $\square_n^+$*-complex projective variety* is defined to be a $\square_n^+$-object of the category of complex projective varieties $(\mathrm{Proj}/\mathbb{C})$. It is also called an *augmented n-cubic complex projective variety*. For $\square_n^+$-complex projective varieties $X_\bullet, Y_\bullet$, we define a morphism $\Phi_\bullet : X_\bullet \to Y_\bullet$ to be a natural transformation from the functor $X_\bullet$ to the one $Y_\bullet$ over the identity functor $\mathrm{id} : \square_n^+ \to \square_n^+$.

1.6 Definition. For a $\square_n^+$-complex projective variety $X_\bullet$, a contravariant functor $Y_\bullet$ from $\square_1^+$ to the category of $\square_n^+$-complex projective varieties is called a *2-resolution of $X_\bullet$* if $Y_\bullet$ is defined by a cartesian square of morphisms of $\square_n^+$-complex projective varieties

$$\begin{array}{ccc} Y_{11\bullet} & \longrightarrow & Y_{01\bullet} \\ \downarrow & & \downarrow f \\ Y_{10\bullet} & \longrightarrow & Y_{00\bullet}, \end{array} \tag{1.1}$$

which satisfies the following conditions:

(i) $Y_{00\bullet} = X_\bullet$,
(ii) $Y_{01\bullet}$ is a smooth $\square_n^+$-complex projective variety, i.e., a contravariant functor from $\square_n^+$ to the category of smooth complex projective varieties,
(iii) the horizontal arrows are closed immersions of $\square_n^+$-complex projective varieties,
(iv) f is a proper morphism between $\square_n^+$-complex projective varieties, and
(v) f induces an isomorphism from $Y_{01\beta} - Y_{11\beta}$ to $Y_{00\beta} - Y_{10\beta}$ for any $\beta \in \mathrm{Ob}(\square_n^+)$.

We think of the cartesian square in (1.1) as a morphism from the $\square_{n+1}^+$-complex projective variety $Y_{1\bullet\bullet}$ to the one $Y_{0\bullet\bullet}$ and write it as $Y_{1\bullet\bullet} \to Y_{0\bullet\bullet}$.

For a 2-resolution $Z_\bullet$ of $Y_{1\bullet\bullet}$, we define the $\square^+_{n+3}$-complex projective variety $rd(Y_\bullet, Z_\bullet)$ by

$$rd(Y_\bullet, Z_\bullet) := \begin{array}{ccc} Z_{11\bullet} & \longrightarrow & Z_{01\bullet} \\ \downarrow & & \downarrow \\ Z_{10\bullet} & \longrightarrow & Y_{0\bullet\bullet}. \end{array}$$

and call it the *reduction* of $\{Y_\bullet, Z_\bullet\}$.

1.7 Definition. Let X be a complex projective variety and let $\{X^1_\bullet, X^2_\bullet, \cdots, X^n_\bullet\}$ be a sequence of $\square^+_r$-complex projective varieties $X^r_\bullet$ $(1 \le r \le n)$ such that

(i) $X^1_\bullet$ is a 2-resolution of X,

(ii) $X^{r+1}_\bullet$ is a 2-resolution of $X^r_{1\bullet}$ for every r with $1 \le r \le n-1$.

Then, by induction on n, we define

$$Z_\bullet := rd(X^1_\bullet, X^2_\bullet, \cdots, X^n_\bullet) := rd(rd(X^1_\bullet, X^2_\bullet, \cdots, X^{n-1}_\bullet), X^n_\bullet).$$

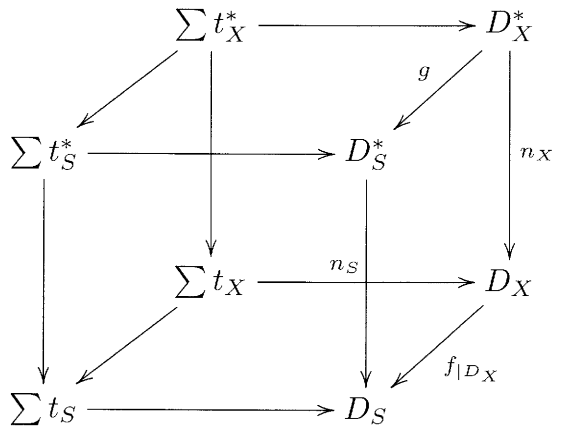
With this notation, if Z_α are smooth for all $\alpha \in \mathrm{Ob}(\square_n)$, we call $Z_\bullet$ an *augmented n-cubic hyper-resolution* of X.

Now we are going to construct a cubic hyper-resolution of an algebraic surface S with ordinary singularities. Let $f : X \longrightarrow S$ be the normalization. We put $D_X := f^{-1}(D_S)$ and $\Sigma t_X := f^{-1}(\Sigma t_S)$. D_X is a singular curve with nodes and Σt_X coincides with the set of nodes of D_X. Since the normal medel X of S is non-singular, a *2-resolution* of S in the sense of F. Guillén, V. Navarro Aznar et al. ([13]) is obtained as follows:

$$(1.2) \qquad \begin{array}{ccc} D_X & \longrightarrow & X \\ {\scriptstyle f_{|D_X}}\downarrow & & \downarrow{\scriptstyle f} \\ D_S & \longrightarrow & S, \end{array}$$

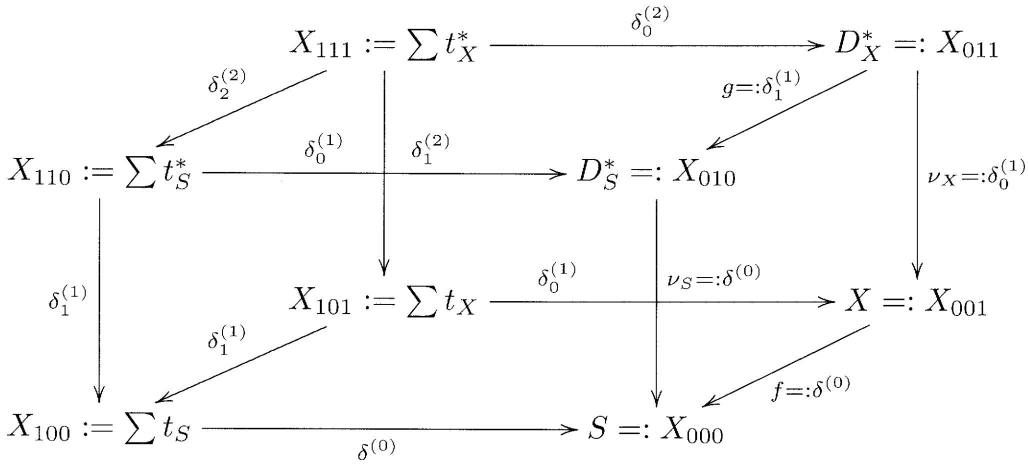
where $f_{|D_X}$ denotes the restriction of f to D_X, and horizontal arrows are inclusion maps. We consider the map $f_{|D_X} : D_X \longrightarrow D_S$ as a $\square^+_0$*-complex projective variety*. Since both of the normal models D^*_S and D^*_X of D_S and D_X, respectively, are non-singular, a 2-resolution of the 0-cubic complex projective variety $f_{|D_X} : D_X \longrightarrow D_S$ in the sense of F. Guillén, V. Navarro Aznar et al. is obtained as follows:

(1.3)



where $n_S : D_S^* \longrightarrow D_S$ and $n_X : D_X^* \longrightarrow D_X$ are the normalizations, $g : D_X^* \longrightarrow D_S^*$ the lifting of the map $f_{|D_X} : D_X \longrightarrow D_S$, and $\Sigma t_S^* := n_S^{-1}(\Sigma t_S)$, $\Sigma t_X^* := n_X^{-1}(\Sigma t_X)$. Replacing $f_{|D_X} : D_X \longrightarrow D_S$ in (1.3) by $f : X \longrightarrow S$ in (1.2), we obtain the following *cubic hyper-resolution* of S in the sense of F. Guillén, V. Navarro Aznar et al.:

(1.4)



where ν_S and ν_X are the composites of the normalizations $n_S : D_S^* \to D_S$ and $n_X : D_X^* \to D_X$ and the inclusion maps $D_S \hookrightarrow S$ and $D_X \hookrightarrow X$, respectively. We put $X_0 := X_{001} \coprod X_{010} \coprod X_{100} = X \coprod D_S^* \coprod \Sigma t_S$ (disjoint union), $X_1 := X_{011} \coprod X_{101} \coprod X_{110} = D_X^* \coprod \Sigma t_X, \coprod \Sigma t_S^*$, $X_2 := X_{111} = \Sigma t_X^*$, $\pi_2 := \delta^{(0)} \circ \delta_{i_1}^{(1)} \circ \delta_{i_2}^{(2)} : X_2 \to S$, $\pi_1 := \delta^{(0)} \circ \delta_{i_1}^{(1)} : X_1 \to S$, and $\pi_0 := \delta^{(0)} : X_0 \to S$, where $i_1 = 0, 1$ and $i_2 = 0, 1, 2$. Then the *semi-simplicial hyper-resolution* of S associated to this cubic hyper-resolution is as follows (cf. [12]):

$$X_2 \begin{array}{c} \xrightarrow{\delta_0^{(2)}} \\ \xrightarrow{\delta_1^{(2)}} \\ \xrightarrow[\delta_2^{(2)}]{} \end{array} X_1 \begin{array}{c} \xrightarrow{\delta_0^{(1)}} \\ \xrightarrow{\delta_1^{(1)}} \end{array} X_0 \xrightarrow{\delta^{(0)}} S$$

We denote symbolically this semi-simplicial hyper-resolution of S by $\pi_\bullet : X._\bullet \rightarrow S$. We denote by $D^+(S, \mathbb{Z})$ the derived category of lower bounded complexes of sheaves of $\mathbb{Z}$-modules over S. We define $K \in \mathrm{Ob}(D^+(S, \mathbb{Z}))$ by

$$K : \quad 0 \rightarrow \pi_{0*}\mathbb{Z}_{X_0} \xrightarrow{d^0} \pi_{1*}\mathbb{Z}_{X_1} \xrightarrow{d^1} \pi_{2*}\mathbb{Z}_{X_2} \rightarrow 0 \quad (K^i = \pi_{i*}\mathbb{Z}_{X_i},\ i = 0, 1, 2)$$

where $d^0 := \delta_0^{(1)*} - \delta_1^{(1)*}$ and $d^1 := \delta_0^{(2)*} - \delta_1^{(2)*} + \delta_2^{(2)*}$. Then $K = \mathbb{Z}_S$ in $D^+(S, \mathbb{Z})$. We define a so-called *weight filtration* W on $K_{\mathbb{Q}} := K \otimes \mathbb{Q} \in \mathrm{Ob}(D^+(S, \mathbb{Q}))$ by $W_{-q}(K_{\mathbb{Q}}) := \sigma_{\cdot \geq q}\pi_{\cdot *}\mathbb{Q}_{X.}$ (stupid filtration). Then $(K_{\mathbb{Q}}, W) \in \mathrm{Ob}(D^+F(S, \mathbb{Q}))$, where $D^+F(S, \mathbb{Q})$ denotes the derived category of filtered, lower bounded complexes of sheaves of $\mathbb{Q}$-modules over S. By calculation we can prove that $K_{\mathbb{C}} := K \otimes \mathbb{C}$ is quasi-isomorphic to $s(\pi_{\bullet *}\Omega^\bullet_{X_\bullet})$, where $\Omega^\bullet_{X_i}$ (i=0,1,2) denotes the holomorphic de Rham complex over X_i and $s(\pi_{\bullet *}\Omega^\bullet_{X_\bullet})$ the simple complex associated to the double complex $\pi_{\bullet *}\Omega^\bullet_{X_\bullet}$. We define a so-called *Hodge filtration* F on $K_{\mathbb{C}} \simeq s(\pi_{\bullet *}\Omega^\bullet_{X_\bullet})$ by $F^p(s(\pi_{\bullet *}\Omega^\bullet_{X_\bullet})) := s(\sigma_{q \geq p}\pi_{\bullet *}\Omega^q_{X_\bullet})$. Then the data:

$$\{\, \mathbb{Z}_S,\ (\pi_{\bullet *}\mathbb{Q}_{X_\bullet}, W),\ \mathbb{Q}_S,\ (s(\pi_{\bullet *}\Omega^\bullet_{X_\bullet}), W, F)\},$$
$$(\ \mathbb{Q}_S \simeq \pi_{\bullet *}\mathbb{Q}_{X_\bullet}, (\pi_{\bullet *}\mathbb{Q}_{X_\bullet}, W) \otimes \mathbb{C} \simeq (s(\pi_{\bullet *}\Omega^\bullet_{X_\bullet}), W)\ )$$

is a *cohomological mixed Hodge complex* in the sense of Deligne. Hence the filtration $W[\ell]$ ($W[\ell]_q := W_{q-\ell}$, the shift of the filtration degree to the right by ℓ) on $H^\ell(S, \mathbb{Q}) \simeq \mathbb{H}^\ell(X_\bullet, \mathbb{Q}_{X_\bullet}) \simeq H^\ell(\mathbb{R}\Gamma(S, s(\pi_{\bullet *}\mathbb{Q}_{X_\bullet})))$ and the filtration F on $H^\ell(S, \mathbb{C}) \simeq \mathbb{H}^\ell(X_\bullet, \mathbb{C}_{X_\bullet}) \simeq \mathbb{H}^\ell(X_\bullet, \Omega^\bullet_{X_\bullet}) \simeq H^\ell(\mathbb{R}\Gamma(S, s(\pi_{\bullet *}\Omega^\bullet_{X_\bullet})))$ $(0 \leq \ell \leq 4)$ define a mixed Hodge structure on $H^\ell(S, \mathbb{Z})$ (modulo torsion). Since the spectral sequence with respect to the weight fltration $W[\ell]$ abuting to $H^\ell(S, \mathbb{Q})$ degenerates at E_2-term (cf. [3], [4]), we have:

1.8 Proposition. *For every pair of integers (ℓ, k) with $0 \leq \ell \leq 4$ and $0 \leq k \leq \ell$,*

$$Gr_k^{W[\ell]} H^\ell(S, \mathbb{Q}) \simeq \frac{Ker\{H^k(X_{\ell-k}, \mathbb{Q}) \xrightarrow{d^{\ell-k}} H^k(X_{\ell-k+1}, \mathbb{Q})\}}{Im\{H^k(X_{\ell-k-1}, \mathbb{Q}) \xrightarrow{d^{\ell-k-1}} H^k(X_{\ell-k}, \mathbb{Q})\}}.$$

where $Gr_k^{W[\ell]} H^\ell(S, \mathbb{Q}) := W[\ell]_k H^\ell(S, \mathbb{Q}) / W[\ell]_{k-1} H^\ell(S, \mathbb{Q})$.

§2 Variation of mixed Hodge structures arising from a locally trivial analytic family of algebraic surfaces with ordinary singularities

We consider a *locally trivial* analytic family of algebraic surfaces with ordinary singularities, from which there arises naturally a variation of mixed Hodge structures.

2.1 Definition. A *locally trivial* analytic family of algebraic surfaces (resp. compact complex surfaces) with ordinary singularities, parametrized by a complex space M, is defined to be a triplet $(\mathfrak{S}, \pi, M)$ such that;

(i) $\pi : \mathfrak{S} \longrightarrow M$ is a surjective holomorphic map between complex spaces,

(ii) $S_t := \pi^{-1}(t)$ is an algebraic surface (resp. compact complex surfaces) with ordinary singularities for every point $t \in M$,

(iii) for every point $p \in \mathfrak{S}$, there exist open neighborhoods $\mathcal{U}$ of p in $\mathfrak{S}$, V of $\pi(p)$ in M with $\pi(\mathcal{U}) = V$, and a biholomorphic map $\phi : \mathcal{U} \longrightarrow U \times V$, where we define $U := \mathcal{U} \cap S_{\pi(p)}$, such that;

(a) the diagram

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\quad\phi\quad} & U \times V \\ & {\scriptstyle \pi_{|\mathcal{U}}} \searrow \quad \swarrow {\scriptstyle Pr_V} & \\ & V & \end{array}$$

commuts,

(b) $\phi_{|U \times p} := id_{U \times p}$.

By taking cubic hyper-resolution of each fiber of the family $\pi : \mathfrak{S} \longrightarrow M$ simultaneously, we obtain a *2-cubic hyper-equisingular family of complex projective varieties, paramerized by* M (See Definition 2.4 later). We denote it by $\mathfrak{X}_\bullet \xrightarrow{a_\bullet} \mathfrak{S} \xrightarrow{\pi} M$.

2.2 Theorem. *Let $(\mathfrak{S}, \pi, M)$ be a locally trivial family of algebraic surfaces with ordinary singularities, parametrized by a complex manifold M. We define $R^\ell_{\mathbb{Z}}(\pi) := R^\ell\pi_*\mathbb{Z}_{\mathfrak{S}}$ modulo torsion $(0 \leq \ell \leq 4)$, $R^\ell_{\mathbb{Q}}(\pi) := R^\ell_{\mathbb{Z}}(\pi) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $R^\ell_{\mathcal{O}}(\pi) := R^\ell\pi_*(\pi^{\cdot}\mathcal{O}_M) \simeq \mathbb{R}^\ell\pi_*(DR^\bullet_{\mathfrak{S}/M})$, where $\pi^{\cdot}\mathcal{O}_M$ is the topological inverse of the structure sheaf of M by the map $\pi : \mathfrak{S} \longrightarrow M$ and $DR^\bullet_{\mathfrak{S}/M}$ the relative cohomological de Rham complex of the family $\pi : \mathfrak{S} \longrightarrow M$ (cf. Theorem 2.9). Then there exist a family of increasing sub-local systems $\mathbb{W}$ (weight filtration) on $R^\ell_{\mathbb{Q}}(\pi)$ and a family of decreasing holomorphic subbundles $\mathbb{F}$ (Hodge filtration) on $R^\ell_{\mathcal{O}}(\pi)$ such that $\{R^\ell_{\mathbb{Z}}(\pi), (R^\ell_{\mathbb{Q}}(\pi), \mathbb{W}[\ell]), (R^\ell_{\mathcal{O}}(\pi), \mathbb{W}[\ell], \mathbb{F})\}$ is a variation of mixed Hodge structures , where $\mathbb{W}[\ell]$ denotes the shift of the filtration degree to the right by ℓ, i.e., $\mathbb{W}[\ell]_q := \mathbb{W}_{q-\ell}$. Furthermore, there exist spectral sequences*

$$_WE_1^{p,q} \simeq R^q(\pi \circ a_p)_*\mathbb{Q}_{\mathfrak{X}_p} \Rightarrow {}_WE_\infty^{p,q} = Gr^W_{-p}(R^{p+q}_{\mathbb{Q}}(\pi)),$$
$$_FE_1^{p,q} \simeq \mathbb{R}^q(\pi \circ a_\bullet)_*\Omega^p_{\mathfrak{X}_\bullet/M} \Rightarrow {}_FE_\infty^{p,q} = Gr^p_F(R^{p+q}_{\mathcal{O}}(\pi))$$

with ${}_WE_2^{p,q} = {}_WE_\infty^{p,q}$, ${}_FE_1^{p,q} = {}_FE_\infty^{p,q}$.

In what follows we shall prove this theorem in a more general setting. That is, we prove this thorem for a *cubic hyper-equisingular family of complex projective varieties*. To explain this notion, we prepare some notation. We denote by $\mathcal{F}_M(\mathrm{Proj}/\mathbb{C})$ (resp. $\mathcal{F}_M(\mathrm{An}/\mathbb{C})$) the category of analytic families of complex projective (resp. analytic) varieties, parametrized by a complex space M.

2.3 Definition. We call a $\square_n^+$-object (resp. $\square_n$-object) of $\mathcal{F}_M(\mathrm{Proj}/\mathbb{C})$ (resp. $\mathcal{F}_M(\mathrm{An}/\mathbb{C})$) an *analytic family of augmented n-cubic (resp. n-cubic) complex projective (resp. analytic) varieties, parametrized by a complex space M.*

Let $b_\bullet : X_\bullet \longrightarrow X$ be an augmented n-cubic complex projective (resp. analytic) variety and M a complex space. Then $X_\alpha \times M$ ($\alpha \in \mathrm{Ob}(\square_n)$), $X \times M$, $a_\alpha := b_\alpha \times \mathrm{id}_M : X_\alpha \times M \to X \times M$ and $\pi := \mathrm{Pr}_M : X \times M \longrightarrow M$, the projection to M, consititute an analytic family of augmented n-cubic complex projective (resp. analytic) varieties , parametrized by a complex space M, which we denote by

$$X_\bullet \times M \xrightarrow{a_\bullet := b_\bullet \times \mathrm{id}_M} X \times M \xrightarrow{\pi := \mathrm{Pr}_M} M$$

and call the *product family of augmented n-cubic complex projective (resp. analytic) varieties, parametrized by a complex space M*. Let $\mathfrak{X}_\bullet^+ = \{a_\bullet : \mathfrak{X}_\bullet \longrightarrow \mathfrak{X}\}$ be an analytic family of augmented n-cubic complex projective (resp. analytic) varieties, parametrized by a complex space M. Whenever we wish to express its parameter space M explicitly, we write

$$\mathfrak{X}_\bullet \xrightarrow{a_\bullet} \mathfrak{X} \xrightarrow{\pi} M. \tag{2.1}$$

For $t \in M$, $X_{t\alpha} := (\pi \circ a_\alpha)^{-1}(t)$ ($\alpha \in \mathrm{Ob}(\square_n)$), $X_t := \pi^{-1}(t)$ and $a_{t\alpha} := a_{\alpha|X_{t\alpha}} : X_{t\alpha} \longrightarrow X_t$ constitute an augmented n-cubic complex projective (resp. analytic) variety. We denote it by $a_{t\bullet} : X_{t\bullet} \longrightarrow X_t$ and call it the fiber at $t \in M$ of an analytic family of augmented n-cubic complex projective (resp. analytic) varieties in (2.1). Similarly, for an open subset U of $\mathfrak{X}$, we form an analytic family

$$a_\bullet^{-1}(\mathcal{U}) \xrightarrow{a_{\bullet|a_\bullet^{-1}(\mathcal{U})}} \mathcal{U} \xrightarrow{\pi} \pi(\mathcal{U})$$

of augmented n-cubic *analytic varieties*, parametized by a complex space $\pi(\mathcal{U})$. With these notions, we define an *n-cubic hyper-equisingular family of complex projective varieties, parametrized by a complex space* as follows:

2.4 Definition. Let $\mathfrak{X}_\bullet \xrightarrow{a_\bullet} \mathfrak{X} \xrightarrow{\pi} M$ be a family of augmented n-cubic complex projective varieties, parametrized by a complex space M. We call $\mathfrak{X}_\bullet \xrightarrow{a_\bullet} \mathfrak{X} \xrightarrow{\pi} M$ an *n-cubic hyper-equisingular family of complex projective varieties, parametrized by a complex space M* if it satisfies the following conditions:

(i) for any pont $t \in M$, $a_{t\bullet} : X_{t\bullet} \longrightarrow X_t$ is an augmented n-cubic hyper-resolution of X_t,

(ii)(*analytical local triviality*) for any point $p \in \mathfrak{X}$, there exists an open neighborhood $\mathcal{U}$ of p in $\mathfrak{X}$ such that $a_\bullet^{-1}(\mathcal{U}) \xrightarrow{a_\bullet} \mathcal{U} \xrightarrow{\pi} \pi(\mathcal{U})$ is analytically isomorphic to

$$(a_\bullet^{-1}(\mathcal{U}) \bigcap X_{\pi(p)\bullet}) \times \pi(\mathcal{U}) \longrightarrow (\mathcal{U} \bigcap X_{\pi(p)}) \times \pi(\mathcal{U}) \xrightarrow{\mathrm{Pr}_{\pi(\mathcal{U})}} \pi(\mathcal{U})$$

over the identy map $\mathrm{id}_{\pi(\mathcal{U})} : \pi(\mathcal{U}) \longrightarrow \pi(\mathcal{U})$

2.5 Proposition. *Let $\mathfrak{X}_\bullet \xrightarrow{a_\bullet} \mathfrak{X} \xrightarrow{\pi} M$ be an n-cubic hyper-equisingular family of complex projective varieties, parametrized by a complex manifold M. Then the $\Box_n$-object $\pi_\bullet : \mathfrak{X}_\bullet \longrightarrow M (\pi_\bullet := \pi \circ a_\bullet)$ of smooth families of complex manifolds, parametrized by M is C^∞ trivial at any point of M; that is, for any point $t_0 \in M$, there exist an open neighborhood N of t_0 in M and a diffeomorphism $\Phi_\bullet : (\pi_\bullet^{-1})(N) \longrightarrow X_{t_0\bullet} \times N$ of $\Box_n$-objects of complex manifolds over the identity map $\mathrm{id}_N : N \longrightarrow N$. Furthermore, $\mathfrak{X}_\bullet \xrightarrow{a_\bullet} \mathfrak{X} \xrightarrow{\pi} M$ is topologically trivial at any point of M.*

Proof. Let N_1 be a coordinate neighborhood of t_0 in M with a holomorphic local coordinate system $(t_1, \cdots, t_m)$, and N a relatively compact open subset of N_1 with $\overline{N} \subset N_1$. Let $t_i = x_i + \sqrt{-1}x_{m+i} (1 \leq i \leq m)$ be the expression of t_i in real local coordinate functions x_i, y_i. To prove the proposition it suffices to show that for every $\partial/\partial x_i$ $(1 \leq i \leq 2m)$ and every $\alpha \in \mathrm{Ob}(\Box_n)$ there exists its liftings v_i^α to $\pi_\alpha^{-1}(N)$, i.e., a C^∞ vector field on $\pi_\alpha^{-1}(N)$ with the properties;

$$(2.2) \qquad \begin{aligned} &\text{(i)} \qquad (d\pi_\alpha)(v_i^\alpha) = \pi_\alpha^*\left(\frac{\partial}{\partial x_i}\right), \\ &\text{(ii)} \qquad dE_{\alpha\beta}(v_i^\beta) = E_{\alpha\beta}^*(v_i^\alpha) \end{aligned}$$

in $E_{\alpha\beta}^*\Theta_{\mathfrak{X}_\alpha}$ for every pair (α, β) of elements of $\mathrm{Ob}(\Box_n)$ with $\alpha \leq \beta$ in the category $\Box_n$, where $E_{\alpha\beta} : \mathfrak{X}_\beta \longrightarrow \mathfrak{X}_\alpha$ denotes a holomorphic map corresponding to an arrow $\alpha \longrightarrow \beta$ in $\Box_n$ and $\Theta_{\mathfrak{X}_\alpha}$ the sheaf of germs of holomorphic vector fields on $\mathfrak{X}_\alpha$.

Indeed, if such liftings $\{v_i^\alpha\}_{\alpha \in \mathrm{Ob}(\Box_n)}$ exist, integrating v_i^α, we have a C^∞-trivialization of the family $\pi_\alpha : \mathfrak{X}_\alpha \longrightarrow N$ along the x_i-axis in N for all $\alpha \in \mathrm{Ob}(\Box_n)$ such that those trivializations commute with the maps $E_{\alpha\beta} : \mathfrak{X}_\beta \longrightarrow \mathfrak{X}_\alpha$ for every pair (α, β) of elements of $\mathrm{Ob}(\Box_n)$ with $\alpha \longrightarrow \beta$ in the category $\Box_n$ due to the condition (ii) in (2.2). Arguing inductively on the dimension of M, we finally obtain the trivialization asserted in the proposition (cf. for more precise argument we refer to Theorem 3.3 in [9]). Now we are going to prove the existence of the liftings v_i^α to $\pi_\alpha^{-1}(N)$ of $\partial/\partial x_i$ subject to the conditions in (2.2).

We take open coverings $\mathcal{V} = \{\mathcal{V}_\lambda\}_{\lambda \in \Lambda_0}$ and $\mathcal{V}' = \{\mathcal{V}'_\lambda\}_{\lambda \in \Lambda_0}$ of $\pi^{-1}(\overline{N})$ in $\mathfrak{X}$ that satisfy the following conditions:

for every $\lambda \in \Lambda_0$,

(i) $\overline{\mathcal{V}_\lambda}$ is a compact subset of $\mathcal{V}'_\lambda$,

(ii) there exists an embedding $\varphi_\lambda : \mathcal{V}'_\lambda \longrightarrow \mathbb{C}^{n_\lambda}$, and

(iii) $a_\bullet^{-1}(\mathcal{V}'_\lambda) \xrightarrow{a_\bullet} \mathcal{V}'_\lambda \xrightarrow{\pi} \pi(\mathcal{V}'_\lambda)$ is analytically trivial.

We are allowed to put the condition (iii) due to the *analytically local triviality* of the family $\mathfrak{X}_\bullet \xrightarrow{a_\bullet} \mathfrak{X} \xrightarrow{\pi} M$ (cf. Definition 2.4 (ii)). By this condition there exist liftings $v^\alpha_{\lambda i}$ of $\partial/\partial x_i$ to $a_\alpha^{-1}(\mathcal{V}'_\lambda)$ for every $\alpha \in \mathrm{Ob}(\Box_n)$ and every $\lambda \in \Lambda_0$, subject to the conditions in (2.2). We take a C^∞ partition of unity $\{\rho_\lambda\}_{\lambda\in\Lambda_0}$ on $\mathfrak{X}' := \bigcup_{\lambda\in\Lambda_0} \mathcal{V}'_\lambda$ subordinate to the covering $\mathcal{V} = \{\mathcal{V}_\lambda\}_{\lambda\in\Lambda_0}$, i.e., ρ_λ's are "C^∞ functions" on $\mathfrak{X}' := \bigcup_{\lambda\in\Lambda_0} \mathcal{V}'_\lambda$ satisfying the following conditions:

(i) $0 \leq \rho_\lambda \leq 1$ for $\lambda \in \Lambda_0$,

(ii) $Supp\, \rho_\lambda \subset \mathcal{V}_\lambda$ for $\lambda \in \Lambda_0$,

(iii) $\sum_{\lambda\in\Lambda_0} \rho_\lambda \equiv 1$ on $\mathfrak{X}'$.

Notice that $\mathfrak{X}'$ is a singular space. We use here the term "C^∞ functions" in the sense of that they are locally pull-backs of C^∞ functions on $\mathbb{C}^{n_\lambda}$ via embeddings $\varphi_\lambda : \mathcal{V}'_\lambda \longrightarrow \mathbb{C}^{n_\lambda}$. The existence of C^∞-partition of unity $\{\rho_\lambda\}_{\lambda\in\Lambda_0}$ as above is guaranteed by the fact that the proof of the existence of C^∞-partition of unity subordinate to a countably indexed open covering of a C^∞-manifold is also applicable in our case (cf.[9, Chapter I, Theorem 4.6]). We define

$$v^\alpha_i := \sum_{\lambda\in\Lambda_0} a^*_\alpha(\rho_\lambda) v^\alpha_{\lambda i}$$

for $\alpha \in \mathrm{Ob}(\Box_n)$. Then we can easily check that

$$(d\pi_\alpha)(v^\alpha_i) = \pi^*_\alpha(\frac{\partial}{\partial x_i}) \quad \text{and}$$

$$(dE_{\alpha\beta})(v^\beta_i) = E^*_{\alpha\beta}(v^\alpha_i)$$

for every pair (α, β) of elements of $\mathrm{Ob}(\Box_n)$ with $\alpha \leq \beta$ in the category $\Box_n$.

Finally, we shall show that the C^∞ triviality of the family $\pi_\bullet : \mathfrak{X}_\bullet \longrightarrow M$ implies the topological triviality of the family $\mathfrak{X}_\bullet \xrightarrow{a_\bullet} \mathfrak{X} \xrightarrow{\pi} M$. For a fiber $X_{t\bullet}$ ($t \in M$) of the family $\pi_\bullet : \mathfrak{X}_\bullet \longrightarrow M$, we define an equivalence relation on the topological space $\coprod_{\alpha\in\mathrm{Ob}(\Box_n)} X_{t\alpha}$ (disjoint sum) by

$$p \sim q \text{ iff } p \in X_{t\alpha}, q \in X_{t\beta} \text{ such that } \begin{cases} \alpha \leq \beta & \text{and } e_{\alpha\beta}(q) = p \\ \text{or} \quad \alpha > \beta & \text{and } e_{\beta\alpha}(p) = q, \end{cases}$$

where $e_{\alpha\beta} : X_{t\beta} \longrightarrow X_{t\alpha}$ (resp. $e_{\beta\alpha} : X_{t\alpha} \longrightarrow X_{t\beta}$) is the holomorphic map corresponding to an arrow $\alpha \longrightarrow \beta$(resp. $\beta \longrightarrow \alpha$) in $\Box_n$. Then the natural map from $(\coprod_{\alpha\in \mathrm{Ob}(\square_n)} X_{t\alpha}/\sim)$ (the quotient topological space of $\coprod_{\alpha\in \mathrm{Ob}(\square_n)} X_{t\alpha}$ by the equivalence relation $\sim$ defined above) to X_t gives rise to a homeomorphism between these spaces, because $X_{t\bullet}$ is a cubic hyper-resolution of X_t. Therefore a diffeomorphism between different fibers $X_{t\bullet}$ and $X_{t'\bullet}(t,t'\in M)$ gives rise to a homeomorphism between different fibers $X_{t\bullet}\longrightarrow X_t$ and $X_{t'\bullet}\longrightarrow X_{t'}$ of the family $\mathfrak{X}_\bullet \xrightarrow{a_\bullet} \mathfrak{X} \xrightarrow{\pi} M$.

Q.E.D.

The relative version of *cohomological descent* holds for a cubic hyper-equisingular family of complex projective varieties. In order to state this fact we prepare some notation and terminology. Let $\Phi_\bullet : X_\bullet \longrightarrow X$ be an n-cubic topological space with an augmentation to a topological space X, i.e., $X_\bullet$ is a contravariant functor from the n-cubic category $\square_n$ to the category of topological space (Top) and $\Phi_\bullet$ is a morphism from $X_\bullet$ to $X\times\square_n$ (cf. Definition 1.2, Definition 1.3 and Definition 1.4). For a commutative ring R with identity, an *R-module preshef* $F^\bullet$ on an n-cubic topological space $X_\bullet:\square_n\longrightarrow(\mathrm{Top})$ is defined to be a contravariant functor from the total category $\mathrm{tot}(X_\bullet)$ to the category of R-modules, where we identify a topological space with the category of open subsets of it. We say an R-module presheaf $F^\bullet$ on an n-cubic topological space $X_\bullet$ is an *R-module sheaf* if the presheaves F^α on X_α, defined by $F^\bullet$, are sheaves for all $\alpha\in\square_n$. For $R-$module (pre)sheaves $F^\bullet$ and $G^\bullet$ on $X_\bullet$, a *morphism from $F^\bullet$ to $G^\bullet$* is defined to be a natural transformation from $F^\bullet$ to $G^\bullet$.

We denote by $\mathcal{M}(X_\bullet,R)$ and $\mathcal{M}(X,R)$ the categories of R-module sheaves on $X_\bullet$ and X, respectively, where R is a commutative ring with identity. For an R-module sheaf $\mathcal{F}$ on X we define its inverse image $\Phi^*\mathcal{F}\in\mathcal{M}(X_\bullet,R)$ in a natural way. The functor $\Phi^*_\bullet:\mathcal{M}(X,R)\longrightarrow\mathcal{M}(X_\bullet,R)$ has a right adjoint $\Phi_{\bullet *}:\mathcal{M}(X_\bullet,R)\longrightarrow\mathcal{M}(X,R)$. Since the functor Φ^* is exact, it defines a functor

$$\Phi^*_\bullet : D^+(X,R)\longrightarrow D^+(X_\bullet,R), \tag{2.3}$$

where $D^+(X,R)$ and $D^+(X_\bullet,R)$ denote the derived categories of lower bounded complexes of R-module sheaves on X and $X_\bullet$, respectively. The functor in (2.3) has a right adjoint

$$\mathbb{R}\Phi_{\bullet *}:D^+(X_\bullet,R)\longrightarrow D^+(X,R).$$

For more details we refer to [13, Exposé I].

Now we introduce some general notation. Let $F^\bullet$ be a lower bounded complex of R-module sheaves on an n-cubic topological space $X_\bullet$. We take the factorization

$$X_\bullet \xrightarrow{a_{1\bullet}} X\times\square_n \xrightarrow{a_{2\bullet}} X \tag{2.4}$$

of $a_\bullet: X_\bullet\longrightarrow X$. By definition $a_{1\bullet *}F^\bullet=\{a_{1\alpha *}F^\alpha\}_{\alpha\in\mathrm{Ob}(\square_n)}$, to which we associate a simple complex $s(a_{1\bullet *}F^\bullet)$ of R-module sheaves on X. To explain this we give the definition of an *n-ple complex of an abelian category*. Let A be an abelian category. We denote by $C^+(A)$ the category of lower bounded complexes of A. Let n be an integer≥ 1. We denote by e_i the i-th vector of the canonical basis of $\mathbb{Z}^n$, i.e., $e_i = (0, \cdots, 1, \cdots, 0)$ (1 is at the i-th place) for $1 \leq i \leq n$.

2.6 Definition. With the notation above, an n-ple complex of A consisits of the following entities:

(i) a $\mathbb{Z}^n$-graded object $\{K^\alpha\}_{\alpha\in\mathbb{Z}^n}$ of A, and

(ii) a family $\{d_i\}_{1\leq i\leq n}$ of differentials of $K^{\cdot}$ such that d_i is of defree e_i and they commute each other.

We denote by n-$C^+(A)$ the category of n-ple complexes of an abelian category A.

2.7 Definition. For $K \in n$-$C^+(A)$ its *associated simple complex* $s(K) \in C^+(A)$ is defined to be as follows:

$$s(K)^p := \sum_{\sum p_i = p} K^{p_1\cdots p_n}, \;\; p \in \mathbb{Z} \text{ and}$$

the defferential d of $s(K)$ is defined by

$$d = \sum_{j=1}^{n} (-1)^{\varepsilon_j} d_j \;\text{ on }\; K^{p_1\cdots p_n},$$

where $\varepsilon_j = \sum_{i<j} p_i$.

Let $\mathcal{A}_\bullet$ be a $(\square_n^+)^\circ$-object of lower bounded complexes of R-module sheaves on a topological space, say Y, i.e., a functor $\mathcal{A}_\bullet : (\square_n^+)^\circ \to C^+(Y, R)$, where $C^+(Y, R)$ is the category of lower bounded complexes of R-module sheaves on Y. We denote $\mathcal{A}_\bullet(\alpha) \in C^+(Y, R)$ by $\mathcal{A}_\alpha^\bullet$ for each $\alpha \in \mathrm{Ob}(\square_n^+)$. We associate to such $\mathcal{A}$ an object $K(\mathcal{A}_\bullet)$ of (n+2)-$C^+(Y, R)$, i.e., an (n+2)-ple lower bounded complex of $\mathcal{M}(Y, R)$ as follows:

$$K(\mathcal{A})^{\alpha_0\cdots\alpha_n q} := \begin{cases} \mathcal{A}_\alpha^q & \text{if } \alpha \in \mathrm{Ob}(\square_n^+) \\ 0 & \text{if } \alpha \in \mathbb{Z}^{n+1} - \mathrm{Ob}(\square_n^+); \end{cases}$$

the (i+1)-th differential is the one induced by the morphism $\alpha \to \alpha + e_i$ in $\square_n^+$ for $0 \leq i \leq n$, and (n+2)-th differential is the one of $\mathcal{A}_\alpha^\bullet$. For the sake of simplicity we denote $s(K(\mathcal{A}_\bullet))$ by $s(\mathcal{A}_\bullet)$.

We think of $a_{1\bullet *}F^\bullet = \{a_{1\alpha *}F^\alpha\}_{\alpha\in\mathrm{Ob}(\square_n)}$ as a $(\square_n^+)^\circ$-object of lower bounded complexes of R-module sheaves on X by defining $F^{(0,\cdots,0)} = \{0\}$ for $(0, \cdots, 0) \in \square_n^+$, and form $s(a_{1\bullet *}F^\bullet)$. Then we have

$$\mathbb{R}a_{2\bullet *}(a_{1\bullet *}F^\bullet) \cong s(a_{1\bullet *}F^\bullet)[1]$$

in $D^+(X,R)$, where [1] stands for the shift of the degree of complexes to the left by 1, i.e., $s(a_{1\bullet *}F^\bullet)[1]^p = s(a_{1\bullet *}F^\bullet)^{p+1}$. Then we have

$$\mathbb{R}a_{\bullet *}F^\bullet \cong s(a_{1\bullet *}F^\bullet)[1] \tag{2.5}$$

in $D^+(X,R)$ (for more definite arguments, see [13]). This description of $\mathbb{R}a_{\bullet *}F^\bullet$ is necessary for our arguments in the following.

2.8 Theorem. *(Cohomological descent of R-module sheaves) Let $\mathfrak{X}_\bullet \xrightarrow{a_\bullet} \mathfrak{X} \xrightarrow{\pi} M$ be an n-cubic ($n \geq 1$) hyper-equisingular family of complex projective varieties, parametrized by a complex space M. Then, for an R-module sheaf $\mathcal{A}$ on $\mathfrak{X}$, the adjunction map*

$$\mathcal{A} \longrightarrow \mathbb{R}a_{\bullet *}a_\bullet^*\mathcal{A}$$

is an isomorphism in $D^+(\mathfrak{X},R)$.

For an n-cubic ($n \geq 1$) hyper-equisingular family $\mathfrak{X}_\bullet \xrightarrow{a_\bullet} \mathfrak{X} \xrightarrow{\pi} M$ of comlex projective varieties, parametrized by a complex space M, we denote by $\Omega^\bullet_{\mathfrak{X}_\alpha/M}$ the relative de Rham complex of a smooth family $\pi \cdot a_\alpha : \mathfrak{X}_\alpha \longrightarrow M$ of complex manifolds for each $\alpha \in \mathrm{Ob}(\square_n)$. Then $\Omega^\bullet_{\mathfrak{X}_\bullet/M} := \{\Omega^\bullet_{\mathfrak{X}_\alpha/M}\}_{\alpha\in \mathrm{Ob}(\square_n)}$ is obviously a complex of sheaves of $\mathbb{C}$-vector spaces on a $\square_n$-complex manifold $\mathfrak{X}_\bullet$.

2.9 Theorem. *(Cohomological descent of relative de Rham complexes) Under the same setting as in the preceding theorem, there exists naturally an isomorphism*

$$DR^\bullet_{\mathfrak{X}/M} \simeq \mathbb{R}a_{\bullet *}\Omega^\bullet_{\mathfrak{X}_\bullet/M}$$

in $D^+(\mathfrak{X},\mathbb{C})$, where $DR^\bullet_{\mathfrak{X}/M}$ is the relative cohomological de Rham complex of a locally trivial family $\pi : \mathfrak{X} \longrightarrow M$, i.e., the relative version of a cohomological de Rham complex of a singular variety (cf. [14, p.28, Remark]).

The proofs of these theorems are almost identical with those in the absolute cases, i.e., M is a single point (cf.[13, p.41, Théorèm 6.9], [13, p.61, Théorèm 1.3]), due to the analytical local triviality of the family $\mathfrak{X}_\bullet \xrightarrow{a_\bullet} \mathfrak{X} \xrightarrow{\pi} M$ (cf. [27]).

2.10 Theorem. *Let $\mathfrak{X}_\bullet \xrightarrow{a_\bullet} \mathfrak{X} \xrightarrow{\pi} M$ be an n-cubic ($n \geq 1$) hyper-equisingular family of complex projective varieties, parametrized by a complex manifold M. We define $R^\ell_{\mathbb{Z}}(\pi) := R^\ell\pi_*\mathbb{Z}_{\mathfrak{X}}$ modulo torsion ($0 \leq \ell \leq 2(\dim\mathfrak{X} - \dim M)$), $R^\ell_{\mathbb{Q}}(\pi) := R^\ell_{\mathbb{Z}}(\pi) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $R^\ell_{\mathcal{O}}(\pi) := R^\ell\pi_*(\pi^{\cdot}\mathcal{O}_M) \simeq \mathbb{R}^\ell\pi_*(DR^\bullet_{\mathfrak{X}/M})$, where $\pi^{\cdot}\mathcal{O}_M$ is the topological inverse of the structure sheaf of M by the map $\pi : \mathfrak{X} \longrightarrow M$ and $DR^\bullet_{\mathfrak{X}/M}$ the relative cohomological de Rham complex of the family $\pi : \mathfrak{X} \longrightarrow M$. Then there exist a family of increasing sub-local systems $\mathbb{W}$ (weight filteration) on $R^\ell_{\mathbb{Q}}(\pi)$ and a family of decreasing holomorphic subbundles $\mathbb{F}$ (Hodge filteration) on $R^\ell_{\mathcal{O}}(\pi)$ such that;*

(i) *there are spectral sequences*

$$ {}_W E_1^{p,q} \simeq \oplus_{|\alpha|=p+1} R^q \pi_{\alpha *}\mathbb{Q}_{\mathfrak{X}_\alpha} \Rightarrow {}_W E_\infty^{p,q} = Gr_{-p}^W(R_{\mathbb{Q}}^{p+q}(\pi)), $$
$$ {}_F E_1^{p,q} \simeq \mathbb{R}^q \pi_*(s(a_{1\bullet *}\Omega^p_{\mathfrak{X}_\bullet/M})[1]) \Rightarrow {}_F E_\infty^{p,q} = Gr_F^p(R_{\mathcal{O}}^{p+q}(\pi)) $$

with ${}_W E_2^{p,q} = {}_W E_\infty^{p,q}$, ${}_F E_1^{p,q} = {}_F E_\infty^{p,q}$,

(ii) $(R^\ell_{\mathbb{Z}}(\pi), \mathbb{W}[\ell], \mathbb{F})\}$ *defines mixed Hodge structure at each point* $t \in M$ *, where* $\mathbb{W}[\ell]$ *denotes the shift of the filtration degree to the right by* ℓ*, i.e.,* $\mathbb{W}[\ell]_q := \mathbb{W}_{q-\ell}$*, and*

(iii) *(the Griffiths transversality)*

$\nabla \mathcal{F}^p \subset \Omega_M^1 \otimes \mathcal{F}^{p-1}$,

where ∇ *denotes the Gauss-Manin connection on* $R^\ell_{\mathcal{O}}(\pi)$.

The outline of the proofs of the asertions (i) and (ii) are as follows: By Theorem 2.8, Theorem 2.9 and (2.5), we have an isomorphism

$$ \pi^{\cdot}\mathcal{O}_M \approx DR^\bullet_{\mathfrak{X}/M} \approx s(a_{1\bullet *}\Omega^\bullet_{\mathfrak{X}_\bullet/M})[1] $$

in $D^+(\mathfrak{X}, \mathbb{C})$, where $a_{1\bullet *}\Omega^\bullet_{\mathfrak{X}_\bullet/M}$ is the n-cubic object of complexes of $\mathbb{C}$-vector spaces coming from $\Omega^\bullet_{\mathfrak{X}_\bullet/M}$, and $s(a_{1\bullet *}\Omega^\bullet_{\mathfrak{X}_\bullet/M})$ is its associated single complex. By this ismorphism we have

$$ R^\ell_{\mathcal{O}}(\pi) := R^\ell \pi_*(\pi^{\cdot}\mathcal{O}_M) \simeq \mathbb{R}^\ell \pi_*(s(a_{1\bullet *}\Omega^\bullet_{\mathfrak{X}_\bullet/M})[1]). $$

To compute the hyper-direct image $\mathbb{R}^\ell \pi_*(s(a_{1\bullet *}\Omega^\bullet_{\mathfrak{X}_\bullet/M})[1])$, we shall use the fine resolution $\mathcal{A}^{\bullet\bullet}_{\mathfrak{X}_\bullet/M}$ of $\Omega^\bullet_{\mathfrak{X}_\bullet/M}$, where $\mathcal{A}^{r,s}_{\mathfrak{X}_\alpha/M}$ are the sheaves of relative C^∞ differential forms of type (r,s) on $\mathfrak{X}_\alpha$ ($\alpha \in \mathrm{Ob}(\Box_n)$). Then the natural homomorphism

$$ s(a_{1\bullet *}\Omega^\bullet_{\mathfrak{X}_\bullet/M})[1] \to s(a_{1\bullet *}\mathrm{tot}\,\mathcal{A}^{\bullet\bullet}_{\mathfrak{X}_\bullet/M})[1] $$

is an isomorphism in $D^+(\mathfrak{X}, \mathbb{C})$, where $\mathrm{tot}\,\mathcal{A}^{\bullet\bullet}_{\mathfrak{X}_\alpha/M}$ is the single complex associated to the double complex $\mathcal{A}^{\bullet\bullet}_{\mathfrak{X}_\alpha/M}$ for each $\alpha \in \mathrm{Ob}(\Box_n)$. Since $s(a_{1\bullet *}\mathrm{tot}\,\mathcal{A}^{\bullet\bullet}_{\mathfrak{X}_\bullet/M})[1]$ is π_*-acyclic, we have

$$ R^\ell_{\mathcal{O}}(\pi) \simeq H^\ell(\pi_* s(a_{1\bullet *}\mathrm{tot}\,\mathcal{A}^{\bullet\bullet}_{\mathfrak{X}_\bullet/M})[1]). $$

We define an increasing filtration $\mathbb{W} = \{W_q\}$ and a decreasing one $\mathbb{F} = \{F^p\}$ on the single complex $L := \pi_* s(a_{1\bullet *}\mathrm{tot}\,\mathcal{A}^{\bullet\bullet}_{\mathfrak{X}_\bullet/M})[1]$ by

$$ W_{-q}(\pi_* s(a_{1\bullet *}\mathrm{tot}\,\mathcal{A}^{\bullet\bullet}_{\mathfrak{X}_\bullet/M})[1]) := \sigma_{|\alpha|\geq q+1}\pi_* s(a_{1\alpha *}\mathrm{tot}\,\mathcal{A}^{\bullet\bullet}_{\mathfrak{X}_\alpha/M}) \ \ (q \geq 0) $$

and

$$F^p(\pi_* s(a_{1\bullet *}\mathrm{tot}\,\mathcal{A}^{\bullet\bullet}_{\mathfrak{X}_\bullet/M})[1]) := \sigma_{k\geq p}\pi_* s(a_{1\bullet *}\mathrm{tot}\,\mathcal{A}^{k\bullet}_{\mathfrak{X}_\bullet/M})[1] \ \ (p \geq 0),$$

where

$$\sigma_{|\alpha|\geq q+1}\pi_* s(a_{1\alpha *}\mathrm{tot}\,\mathcal{A}^{\bullet\bullet}_{\mathfrak{X}_\alpha/M}) := \sigma_{\geq q}(L)$$

if we put $L := \pi_* s(a_{1\bullet *}\mathrm{tot}\,\mathcal{A}^{\bullet\bullet}_{\mathfrak{X}_\bullet/M})[1]$. ($\sigma_{\geq q}$: *stupid filtration*). Notice that the filtration W is defined over $\mathbb{Q}$. We calculate the spectral sequence associated to these filtrations, abutting to $R^\ell_{\mathcal{O}}(\pi)$. Since (L_t, W, F) is a *cohomological mixed Hodge complex* in the sense of Deligne for any $t \in M$ (for definition see [3, (8.1.6)]), the spectral sequence $\{E_r(L_t, W), d_r\}$ degenerates at the E_2-terms and the one associated to F at the E_1-terms ([4, p.48, Théorème 3.2.1 (Deligne), (vi), (v)). The assertions (i) and (ii) follow from these facts.

The assertion (iii) can be proved by mimicking the argument of Kats and Oda in [15] for a smooth family of algebraic manifolds. The remaining part of this section will be devoted to the proof of the assertion (iii). We are now going to give an explicit description of the Gauss-Manin connection ∇ on $R^\ell_{\mathcal{O}}(\pi)$ in terms of local coordinates on the cubic hyper-equisingular family $\mathfrak{X}_\bullet \xrightarrow{a_\bullet} \mathfrak{X} \xrightarrow{\pi} M$. We proceed along the arguments in [15], slightly changing them so that they fit to the case of cubic hyper-equisingular families of complex projective varieties. The arguments will proceed in the following three steps:

(I) Definition of an integrable connection $\nabla : R^\ell_{\mathcal{O}}(\pi) \longrightarrow \Omega^1_M \otimes R^\ell_{\mathcal{O}}(\pi)$.

(II) Explicit calculation of the connection.

(III) Proof of $\mathrm{Ker}\nabla = R^n_{\mathbb{C}}(\pi) := R^\ell_{\mathbb{Z}}(\pi) \otimes_{\mathbb{Z}} \mathbb{C}$.

Step (I): *Definition of an integrable connection* $\nabla : R^\ell_{\mathcal{O}}(\pi) \longrightarrow \Omega^1_M \otimes R^\ell_{\mathcal{O}}(\pi)$.

Since the family of algebraic manifolds $\pi_\alpha : \mathfrak{X}_\alpha \longrightarrow M$ is smooth for every $\alpha \in \Box_n$ (n=the length of a cubic hyper-resolution $a_{t\bullet} : X_{t\bullet} \to X_t$ of a fiber of the family $\mathfrak{X}_\bullet \xrightarrow{a_\bullet} \mathfrak{X} \xrightarrow{\pi} M$), the sequence

$$0 \longrightarrow \pi^*_\alpha(\Omega^1_M) \longrightarrow \Omega^1_{\mathfrak{X}_\alpha} \longrightarrow \Omega^1_{\mathfrak{X}_\alpha/M} \longrightarrow 0 \ \ (\alpha \in \Box_n)$$

is exact. $\{\Omega^{\bullet-p}_{\mathfrak{X}_\alpha} \otimes_{\mathcal{O}_{\mathfrak{X}_\alpha}} \pi^*_\alpha\Omega^p_M\}_{\alpha\in\Box_n}$ constitutes a sub-complex of sheaves of $\Omega^\bullet_{\mathfrak{X}_\bullet}$, because

$$E^*_{\alpha\beta}(\Omega^{\bullet-p}_{\mathfrak{X}_\alpha} \otimes_{\mathcal{O}_{\mathfrak{X}_\alpha}} \pi^*_\alpha\Omega^p_M) \subset \Omega^{\bullet-p}_{\mathfrak{X}_\beta} \otimes_{\mathcal{O}_{\mathfrak{X}_\beta}} \pi^*_\beta\Omega^p_M$$

for every integer p with $0 \leq p \leq m$ (m = $\dim M$) and for every pair (α, β) of $\alpha, \beta \in \Box_n$ with $\alpha \longrightarrow \beta$ in the category $\Box_n$, where $E_{\alpha\beta} : \mathfrak{X}_\beta \longrightarrow \mathfrak{X}_\alpha$ is a holomorphic map over M corresponding to an arrow $\alpha \longrightarrow \beta$ in $\Box_n$. Hence the complex $s(a_{1\bullet *}\Omega^\bullet_{\mathfrak{X}_\bullet})$ admits a canonical filtration

$$s(a_{1\bullet *}\Omega^\bullet_{\mathfrak{X}_\bullet}) = G^0(s(a_{1\bullet *}\Omega^\bullet_{\mathfrak{X}_\bullet})) \supset G^1(s(a_{1\bullet *}\Omega^\bullet_{\mathfrak{X}_\bullet})) \supset G^2(s(a_{1\bullet *}\Omega^\bullet_{\mathfrak{X}_\bullet})) \supset \cdots\cdots$$

where

$$\begin{aligned} G^p &= G^p(s(a_{1\bullet *}\Omega^\bullet_{\mathfrak{X}_\bullet})) \\ &= \mathrm{Im}\{s(a_{1\bullet *}(\Omega^{\bullet -p}_{\mathfrak{X}_\bullet} \otimes_{\mathfrak{X}_\bullet} \pi^*_\bullet \Omega^p_M)) \longrightarrow s(a_{1\bullet *}\Omega^\bullet_{\mathfrak{X}_\bullet})\} \end{aligned}$$

The associated graded objects of this filtration are given by

$$\begin{aligned} gr^p = gr^p(s(a_{1\bullet *}\Omega^\bullet_{\mathfrak{X}_\bullet})) &:= G^p/G^{p+1} \\ &\cong s(a_{1\bullet *}(\pi^*_\bullet(\Omega^p_M) \otimes_{\mathcal{O}_{\mathfrak{X}_\bullet}} \Omega^{\bullet -p}_{\mathfrak{X}_\bullet/M})) \end{aligned}$$

Therefore the spectral sequence effected by the filtration $\{G^p\}$ and abutting to the graded objects of $H^\bullet(\mathbb{R}\pi_*(s(a_{1\bullet *}\Omega^\bullet_{\mathfrak{X}_\bullet})))$ with respect to the filtration induced by $\{G^p\}$ is as follows:

$$\begin{aligned} E_1^{p,q} &:= \mathbb{R}^{p+q}\pi_*(gr^p) = \mathbb{R}^{p+q}(s(a_{1\bullet *}(\pi^*_\bullet\Omega^p_M \otimes_{\mathcal{O}_{\mathfrak{X}_\bullet}} \Omega^{\bullet -p}_{\mathfrak{X}_\bullet/M}))) \\ &\simeq \mathbb{R}^q\pi_*(s(a_{1\bullet *}(\pi^*_\bullet\Omega^p_M \otimes_{\mathcal{O}_{\mathfrak{X}_\bullet}} \Omega^\bullet_{\mathfrak{X}_\bullet/M}))) \\ &\simeq \Omega^p_M \otimes_{\mathcal{O}_M} \mathbb{R}^q\pi_*(s(a_{1\bullet *}\Omega^\bullet_{\mathfrak{X}_\bullet/M})) \simeq \Omega^p_M \otimes_{\mathcal{O}_M} R^q_{\mathcal{O}}(\pi) \\ &\Longrightarrow E_\infty^{p,q} = Gr^p(\mathbb{R}^{p+q}\pi_*(s(a_{1\bullet *}\Omega^\bullet_{\mathfrak{X}_\bullet}))) \end{aligned}$$

Since the filtration on $s(a_{1\bullet *}\Omega^\bullet_{\mathfrak{X}_\bullet})$ is compatible with the exterior product $G^i \wedge G^j \subset G^{i+j}$, and since the sequence of functor $\mathbb{R}^q\pi_*$ is multiplicative, the spectral sequence has a product structure; that is, there are pairings

$$E_r^{p,q} \times E_r^{p',q'} \longrightarrow E_r^{p+p',q+q'}$$

for each p, q, p', q' and r, sending (e, e') to $e \cdot e'$ where e, e' are sections of $E_r^{p,q}$ and $E_r^{p',q'}$, respectively, over an open subset of M. This pairing satisfies

$$\begin{aligned} &e \cdot e' = (-1)^{(p+q)(p'+q')} e' \cdot e, \qquad \text{and} \\ &d_r(e \cdot e') = d_r(e) \cdot e' + (-1)^{p+q} e \cdot d_r(e'). \end{aligned}$$

The E_1 terms of the spectral sequence are as follows:

$$0 \longrightarrow R^q_{\mathcal{O}}(\pi) \xrightarrow{d_1^{0,q}} \Omega^1_M \otimes R^q_{\mathcal{O}}(\pi) \xrightarrow{d_1^{1,q}} \Omega^2_M \otimes R^q_{\mathcal{O}}(\pi) \longrightarrow \cdots.$$

In order to show that $d_1^{0,q}$ is a connection, let us consider the pairing

$$E_1^{0,0} \times E_1^{0,q} \longrightarrow E_1^{0,q},$$

which satisfies

$$d_1^{o,q}(\omega \cdot e) = d_1^{0,0}\omega \cdot e + \omega \cdot d_1^{0,q} e, \tag{2.6}$$

where ω, e are sections of $E_1^{0,0} \simeq R^0_{\mathcal{O}}(\pi) \simeq \mathcal{O}_M$ and $E_1^{0,q} = R^q_{\mathcal{O}}(\pi)$, respectively, over an open subset of M. Since

$$d_1^{0,0} : E_1^{0,0} \simeq \mathcal{O}_M \longrightarrow E_1^{1,0} = \Omega^1_M$$

is nothing but the exterior differentiation d_M on M, (2.6) shows that $d_1^{0,q}$ is certainly a connection. Furthermore, since

$$d_1^{1,q} : E_1^{1,q} = \Omega_M^1 \otimes R_{\mathcal{O}}^q(\pi) \longrightarrow E_1^{2,q} = \Omega_M^2 \otimes R_{\mathcal{O}}^q(\pi)$$

is equal to $d_M \otimes 1$, $d_1^{1,q} \cdot d_1^{0,q} = 0$ shows that $d_1^{0,q}$ is an integrable connection, which is nothing but the Gauss-Manin connection on $R_{\mathcal{O}}^q(\pi)$, so we denote $d_1^{o,q}$ by ∇ in what follows.

Step(II): *Explicit calculation of the connection.*

First, notice that, in the spectral sequence of a filtered object, the differential

$$d_1^{p,q} : E_1^{p,q} = \mathbb{R}^{p+q}\pi_*(gr^p) \longrightarrow E_1^{p+1,q} = \mathbb{R}^{p+q+1}\pi_*(gr^{p+1})$$

is the connecting homomorphism of the functor $\mathbb{R}^q\pi_*$ for the exact sequence

$$0 \longrightarrow gr^{p+1} \longrightarrow G^p/G^{p+2} \longrightarrow gr^p \longrightarrow 0$$

Using this fact, we shall explicitly calculate the connection

$$\begin{aligned} d_1^{0,q} : E_1^{0,q} &= \mathbb{R}^q\pi_*(gr^0) \simeq R_{\mathcal{O}}^q(\pi) \\ &\longrightarrow E_1^{1,q} = \mathbb{R}^{q+1}\pi_*(gr^1) \simeq \Omega_M^1 \otimes R_{\mathcal{O}}^q(\pi). \end{aligned}$$

For a cubic hyper-equisingular family $\mathfrak{X}_\bullet \xrightarrow{a_\bullet} \mathfrak{X} \xrightarrow{\pi} M$, we take a point $o \in M$ and put

$$X_\alpha := (\pi \circ a_\alpha)^{-1}(o), \;\; X := \pi^{-1}(o).$$

By the definition of an n-cubic hyper-equisingular family $\mathfrak{X}_\bullet \xrightarrow{a_\bullet} \mathfrak{X} \xrightarrow{\pi} M$, it is *analytically locally trivial*. Hence, schrinking M sufficiently small around o, we may assume that there is a system of open coverings $\mathcal{U}_\alpha := \{U_i^{(\alpha)}\}_{i \in \Lambda_\alpha}$ of X_α (Λ_α: finite set, $\alpha \in \square_n^+$) consisting of Stein coordinate neighborhoods, which is subject to the following requirements;

(2.7)

(i) for each pair (α, β) of elements of $\mathrm{Ob}(\square_n^+)$ with $\alpha \longrightarrow \beta$ in $\square_n^+$, there is a map $\lambda_{\alpha\beta} : \Lambda_\beta \longrightarrow \Lambda_\alpha$ such that:

  (a) if α, β, γ are elements of $\mathrm{Ob}(\square_n^+)$ with $\alpha \longrightarrow \beta \longrightarrow \gamma$ in $\square_n^+$, then $\lambda_{\alpha\gamma} = \lambda_{\alpha\beta} \cdot \lambda_{\beta\gamma}$, and

  (b) $e_{\alpha\beta}(U_i^{(\beta)}) \subset U_{\lambda_{\alpha\beta}(i)}^{(\alpha)}$ for any $i \in \Lambda_\beta$, where $e_{\alpha\beta} : X_\beta \longrightarrow X_\alpha$ is a holomorphic map corresponding to an arrow $\alpha \longrightarrow \beta$ in $\square_n^+$,

(ii) if we define $V_i^{(\alpha)} := U_i^{(\alpha)} \times M$ for $\alpha \in \mathrm{Ob}(\square_n^+)$ and $i \in \Lambda_\alpha$, then $\mathcal{V}_\alpha := \{V_i^{(\alpha)}\}$ is a Stein covering of $\mathfrak{X}_\alpha$ for every $\alpha \in \mathrm{Ob}(\square_n^+)$, and

(iii) $E_{\alpha\beta | V_i^\beta} : V_i^{(\beta)} \longrightarrow V_{\lambda_{\alpha\beta}(i)}^{(\alpha)}$ is equal to $e_{\alpha\beta | U_i^{(\beta)}} \times \mathrm{id}_M$, where $E_{\alpha\beta} : \mathfrak{X}_\beta \longrightarrow \mathfrak{X}_\alpha$ is a holomorphic map over M corresponding to an arrow

$\alpha \longrightarrow \beta$ in $\square_n^+$ for $\alpha, \beta \in \mathrm{Ob}(\square_n^+)$ and $i \in \Lambda_\alpha$,
(iv) $\pi_{\alpha|V_i^{(\alpha)}} = \mathrm{Pr}_M : V_i^{(\alpha)} := U_i^{(\alpha)} \times M \longrightarrow M$ (the projection to M), where $\pi_\alpha := \pi \circ a_\alpha$ and $\pi_0 = \pi$.

Notice that, in order to prove the existence of such a system of open coverings of an n-cubic hyper-equisingular family $\mathfrak{X}_\bullet \xrightarrow{a_\bullet} \mathfrak{X} \xrightarrow{\pi} M$ consisting of Stein coordinate neighborhoods , we use the fact that, for a holomorphic map $f : X \longrightarrow Y$ between complex spaces, the intersection $f^{-1}(U) \cap V$ of the inverse image of a Stein subset U of Y by f and a Stein subset V of X is Stein (cf. [11, p.33, Chapter I, §4, 4]). Schrinking M sufficiently small around o, we take such a system of open coverings of Stein coordinate neighborhoods of $\mathfrak{X}_\bullet \xrightarrow{a_\bullet} \mathfrak{X} \xrightarrow{\pi} M$ and fix it. Using this special covering of $\mathfrak{X}_\bullet$, we shall explicitly describe the map

$$\begin{aligned}\nabla : \mathbb{H}^\ell(M, \pi_*(s(a_{1\bullet *}\Omega^\bullet_{\mathfrak{X}_\bullet/M})[1])) \\ \longrightarrow \Gamma(M, \Omega^1_M) \otimes_{\Gamma(M,\mathcal{O}_M)} \mathbb{H}^\ell(M, \pi_*(s(a_{1\bullet *}\Omega^\bullet_{\mathfrak{X}_\bullet/M})[1])).\end{aligned}$$

In what follows we shall always calculate under this setting unless otherwise mentioned.

Let $\{\mathcal{K}^\bullet_\alpha, d_\alpha\}_{\alpha \in \square_n}$ be a bounded complex of sheaves of coherent analytic sheaves on $\mathfrak{X}_\bullet$. Let $\{\mathfrak{C}^\bullet(\mathcal{V}_\alpha, \mathcal{K}^p_\alpha), \delta_\alpha\}$ be the Cěch complex consisting of alternating cochains with values in $\mathcal{K}^p_\alpha$, respecting the Stein covering $\mathcal{V}_\alpha$; that is,

$$\mathfrak{C}^q(\mathcal{V}_\alpha, \mathcal{K}^p_\alpha) = \oplus_{(i_0 \cdots i_q) \in \Lambda_\alpha^{q+1}} \Gamma(V^{(\alpha)}_{i_0} \cap \cdots \cap V^{(\alpha)}_{i_q}, \mathcal{K}^p_\alpha)$$

and the coboundary map $\delta_\alpha^{(p,q)} : \mathfrak{C}^q(\mathcal{V}_\alpha, \mathcal{K}^p_\alpha) \longrightarrow \mathfrak{C}^{q+1}(\mathcal{V}_\alpha, \mathcal{K}^p_\alpha)$ is defined by

$$(\delta_\alpha^{(p,q)}\beta)(\alpha; p; i_0 \cdots i_{q+1}) = (-1)^p \sum_{j=0}^{q+1} (-1)^j \beta(i_0 \cdots \check{i_j} \cdots i_{q+1})$$

for $\beta = \{\beta(\alpha; p; i'_0 \cdots i'_q)\} \in \mathfrak{C}^q(\mathcal{V}_\alpha, \mathcal{K}^p_\alpha)$, where $\beta(\alpha; p; i'_0 \cdots i'_q) \in \Gamma(V^{(\alpha)}_{i'_0} \cap \cdots \cap V^{(\alpha)}_{i'_q}, \mathcal{K}^p_\alpha)$ for $(i'_0, \cdots, i'_q) \in \Lambda_\alpha^{q+1}$. The pre-sheaf

$$V \longmapsto \mathfrak{C}^q(\mathcal{V}_\alpha \cap V, \mathcal{K}^p_\alpha) := \oplus_{(i_0 \cdots i_q) \in \Lambda_\alpha^{q+1}} \Gamma(V^{(\alpha)}_{i_0} \cap \cdots \cap V^{(\alpha)}_{i_q} \cap V, \mathcal{K}^p_\alpha),$$

defines a sheaf where V is an open subset of $\mathfrak{X}_\alpha$, which we denote by $\mathcal{C}^q(\mathcal{V}_\alpha, \mathcal{K}^p_\alpha)$. We associate to the double complex of abelian sheaves $\mathcal{C}^\bullet(\mathcal{V}_\alpha, \mathcal{K}^\bullet_\alpha)$ a single complex $\mathrm{tot}\, \mathcal{C}^\bullet(\mathcal{V}_\alpha, \mathcal{K}^\bullet_\alpha)$ defined as follows:

$$\begin{aligned}&(\mathrm{tot}\, \mathcal{C}^\bullet(\mathcal{V}_\alpha, \mathcal{K}^\bullet_\alpha))^r := \oplus_{p+q=r} \mathcal{C}^q(\mathcal{V}_\alpha, \mathcal{K}^p_\alpha), \\ &d_\alpha^{(r)} := \oplus_{p+q=r} (-1)^{|\alpha|} (d'^p_\alpha + \delta^{p,q}_\alpha) : (\mathrm{tot}\, \mathcal{C}^\bullet(\mathcal{V}_\alpha, \mathcal{K}^\bullet_\alpha))^r \longrightarrow (\mathrm{tot}\, \mathcal{C}^\bullet(\mathcal{V}_\alpha, \mathcal{K}^\bullet_\alpha))^{r+1},\end{aligned}$$

where $\delta_\alpha^{(p,q)}$ is the Cěch coboundary map $\mathcal{C}^q(\mathcal{V}_\alpha, \mathcal{K}^p_\alpha) \to \mathcal{C}^{q+1}(\mathcal{V}_\alpha, \mathcal{K}^p_\alpha)$ and d'^p_α is the map $\mathcal{C}^q(\mathcal{V}_\alpha, \mathcal{K}^p_\alpha) \to \mathcal{C}^q(\mathcal{V}_\alpha, \mathcal{K}^{p+1}_\alpha)$ induced by the differential of the complex $\mathcal{K}^\bullet_\alpha$ and $|\alpha| = \alpha_0 + \cdots + \alpha_n$ for $\alpha \in \Box_n$. Obviously, $\{\mathrm{tot}\,\mathcal{C}^\bullet(\mathcal{V}_\alpha, \mathcal{K}^\bullet_\alpha), d_\alpha\}_{\alpha\in\Box_n}$ defines a complex of abelian sheaves on $\mathfrak{X}$. which is quasi-isomorphic to $\{\mathcal{K}^\bullet_\alpha, d'_\alpha\}_{\alpha\in\Box_n}$, because $\{\mathrm{tot}\,\mathcal{C}^\bullet(\mathcal{V}_\alpha, \mathcal{K}^\bullet_\alpha), d_\alpha\}$ is quasi-isomorphic to $(\mathcal{K}^\bullet_\alpha, d_\alpha)$ for every $\alpha \in \Box_n$.

2.11 Proposition. *The single complex of abelian sheaves* $s(a_{1\bullet *}\mathrm{tot}\,\mathcal{C}^\bullet(\mathcal{V}_\bullet, \mathcal{K}^\bullet_\bullet)))$ *is* π_**-acyclic. Hence*

$$\begin{aligned}\mathbb{H}^k(M, \pi_* s(a_{1\bullet *}\mathcal{K}^\bullet_\bullet)[1]) &\simeq \mathbb{H}^k(\mathfrak{X}, s(a_{1\bullet *}\mathcal{K}^\bullet_\bullet)[1]) \\ &\simeq \mathbb{H}^k(\mathfrak{X}, s(a_{1\bullet *}\mathrm{tot}\,\mathcal{C}^\bullet(\mathcal{V}_\alpha, \mathcal{K}^\bullet_\alpha))[1]) \\ &\simeq H^k(s(\oplus_{\alpha\in\Box_n}\Gamma(\mathfrak{X}_\alpha, \mathrm{tot}\,\mathcal{C}^\bullet(\mathcal{V}_\alpha, \mathcal{K}^\bullet_\alpha))[1]) \\ &\simeq H^k(s(\mathrm{tot}\,\mathfrak{C}^\bullet(\mathcal{V}_\bullet, \mathcal{K}^\bullet_\bullet))[1]) \ \textit{for}\ k \geq 0,\end{aligned}$$

where $s(\mathrm{tot}\,(\mathfrak{C}^\bullet(\mathcal{V}_\bullet, K^\bullet_\bullet))))$ *is the single complex of abelian groups associated to the (n+2)-ple complex* $\mathrm{tot}\,\mathfrak{C}^\bullet(\mathcal{V}_\bullet, \mathcal{K}^\bullet_\bullet)$.

Proof. In order to prove that $s(a_{1\bullet *}\mathrm{tot}\,\mathcal{C}^\bullet(\mathcal{V}_\bullet, \mathcal{K}^\bullet_\bullet))[1]$ is π_*-acyclic, it suffices to show that

$$H^k(\pi^{-1}(U), (s(a_{1\bullet *}\mathrm{tot}\,\mathcal{C}^\bullet(\mathcal{V}_\bullet, \mathcal{K}^\bullet_\bullet))[1])^r) = 0 \ \ (k \geq 1, r \in \mathbb{Z}) \tag{2.8}$$

for a sufficiently small open subset U of M. Let us notice that

$$(s(a_{1\bullet *}\mathrm{tot}\,\mathcal{C}^\bullet(\mathcal{V}_\bullet, \mathcal{K}^\bullet_\bullet))[1])^r = \oplus_{|\alpha|+p+q=r+1} a_{\alpha *}\mathcal{C}^q(\mathcal{V}_\alpha, \mathcal{K}^p_\alpha),$$

hence

$$\begin{aligned}&H^k(\pi^{-1}(U), (s(a_{1\bullet *}\mathrm{tot}\,\mathcal{C}^\bullet(\mathcal{V}_\bullet, \mathcal{K}^\bullet_\bullet))[1])^r) \\ &= \oplus_{|\alpha|+p+q=r+1} H^k(\pi^{-1}(U), a_{\alpha *}\mathcal{C}^q(\mathcal{V}_\alpha, \mathcal{K}^p_\alpha)).\end{aligned} \tag{2.9}$$

Concerning the holomorphic map $a_{\alpha|\pi_\alpha^{-1}(U)} : \pi_\alpha^{-1}(U) \to \pi^{-1}(U)$, where $a_\alpha : \mathfrak{X}_\alpha \to \mathfrak{X}$ and $\pi_\alpha := \pi \cdot a_\alpha : \mathfrak{X}_\alpha \to M$ are the same ones as in (2.7), we have the Leray spectral sequence

$$\begin{aligned}E_2^{s,k-s} &= H^k(\pi^{-1}(U), R^s a_{\alpha *}\mathcal{C}^q(\mathcal{V}_\alpha, \mathcal{K}^p_\alpha))) \\ &\to E_\infty^k = H^k(\pi_\alpha^{-1}(U), \mathcal{C}^q(\mathcal{V}_\alpha, \mathcal{K}^p_\alpha)).\end{aligned}$$

From this it follows that

$$H^k(\pi^{-1}(U), a_{\alpha *}\mathcal{C}^q(\mathcal{V}_\alpha, \mathcal{K}^p_\alpha)) \simeq H^k(\pi_\alpha^{-1}(U), \mathcal{C}^q(\mathcal{V}_\alpha, \mathcal{K}^p_\alpha)),$$

since $R^s a_{\alpha *}\mathcal{C}^q(\mathcal{V}_\alpha, \mathcal{K}^p_\alpha) = 0$ for $s \geq 1$. On the other hand,

$$H^k(\pi_\alpha^{-1}(U), \mathcal{C}^q(\mathcal{V}_\alpha, \mathcal{K}^p_\alpha)) = 0 \ \text{ for } k \geq 1;$$

hence

$$H^k(\pi^{-1}(U), a_{\alpha *}\mathcal{C}^q(\mathcal{V}_\alpha, \mathcal{K}^p_\alpha)) = 0 \text{ for } k \geq 1$$

Consequently, by (2.9), we obtain (2.8). The latter part of the proposition follows from the facts that the natural map $s(a_{1\bullet *}\mathcal{K}^\bullet_\bullet)[1] \to s(a_{1\bullet *}\mathrm{tot}\,(\mathcal{C}^\bullet(\mathcal{V}_\bullet, \mathcal{K}^\bullet_\bullet))[1]$ is an isomorphism in $D^+(\mathfrak{X}, \underline{Ab})$ where $\underline{Ab}$ denotes the category of abelian sheaves on $\mathfrak{X}$, and that

$$\begin{aligned}\mathbb{H}^k(\mathfrak{X}, s(a_{1\bullet *}\mathcal{K}^\bullet_\bullet)[1]) &\simeq \mathbb{H}^k(\mathfrak{X}, s(a_{1\bullet *}\mathrm{tot}\,\mathcal{C}^\bullet(\mathcal{V}_\bullet, \mathcal{K}^\bullet_\bullet))[1]) \\ &\simeq H^k(\mathfrak{X}, \Gamma(\mathfrak{X}, s(a_{1\bullet *}\mathrm{tot}\,\mathcal{C}^\bullet(\mathcal{V}_\bullet, \mathcal{K}^\bullet_\bullet))[1])) \\ &\simeq \mathbb{H}^k(s(\mathrm{tot}\ \mathfrak{C}^\bullet(\mathcal{V}_\bullet, \mathcal{K}^\bullet_\bullet))[1]).\end{aligned}$$

Q.E.D.

By Proposition 2.11 the explicit description of

$$(2.10)\quad \begin{aligned}&\nabla : \mathbb{H}^q(M, \pi_* s(a_{1\bullet *}\Omega^\bullet_{\mathfrak{X}_\bullet/M})[1]) \\ &\quad \to \Gamma(M, \Omega^1_M) \otimes_{\Gamma(M, \mathcal{O}_M)} \mathbb{H}^q(M, \pi_* s(a_{1\bullet *}\Omega^\bullet_{\mathfrak{X}_\bullet/M})[1])\end{aligned}$$

is reduced to that of

$$\begin{aligned}&\nabla : H^q(s(\mathrm{tot}\,\mathfrak{C}^\bullet(\mathcal{V}_\bullet, \Omega^\bullet_{\mathfrak{X}_\bullet/M}))[1]) \\ &\quad \to \Gamma(M, \Omega^1_M) \otimes_{\Gamma(M, \mathcal{O}_M)} H^q(s(\mathrm{tot}\,\mathfrak{C}^\bullet(\mathcal{V}_\bullet, \Omega^\bullet_{\mathfrak{X}_\bullet/M}))[1]).\end{aligned}$$

In what follows we shall use the notation

$$K^\bullet(\mathcal{F}^\bullet_\bullet) := s(\oplus_{\alpha \in \Box_n}\mathrm{tot}(\mathfrak{C}^\bullet(\mathcal{V}_\alpha, \mathcal{F}^\bullet_\alpha))$$

for a complex of abelian sheaves $\mathcal{F}^\bullet_\bullet$ on $\mathfrak{X}_\bullet$. With this notation we have the following exact sequences of abelian groups:

$$(2.11)\qquad 0 \to K^\bullet(G^1(\Omega^\bullet_{\mathfrak{X}_\bullet})) \to K^\bullet(\Omega^\bullet_{\mathfrak{X}_\bullet}) \to K^\bullet(\Omega^\bullet_{\mathfrak{X}_\bullet/M}) \to 0,$$

$$(2.12)\qquad 0 \to K^\bullet(Gr^1(\Omega^\bullet_{\mathfrak{X}_\bullet})) \to K^\bullet(\Omega^\bullet_{\mathfrak{X}_\bullet}/G^2(\Omega^\bullet_{\mathfrak{X}_\bullet})) \to K^\bullet(\Omega^\bullet_{\mathfrak{X}_\bullet/M}) \to 0.$$

For $\alpha \in \Box_n$ and $i \in \Lambda_\alpha$, we denote by $(x^{(\alpha)}_{i1}, \cdots, x^{(\alpha)}_{in_\alpha})$ a local coordinate system on $U^{(\alpha)}_i$, where $n_\alpha := \dim U^{(\alpha)}_i$. We denote by $(t_1, \cdots, t_m)$ a local coordinate system on M. We decompose the exterior differentiation $d_{\mathfrak{X}_\alpha}$ on $\mathfrak{X}_\alpha$ as

$$(2.13)\qquad d_{\mathfrak{X}_\alpha} = d_M + d_{U^{(\alpha)}_i}$$

on each $V_i^{(\alpha)} = U_i^{(\alpha)} \times M$, where d_M and $d_{U_i^{(\alpha)}}$ are the differentiations with respect to $(t_1, \cdots, t_m)$ and $(x_{i1}^{(\alpha)}, \cdots, x_{in_\alpha}^{(\alpha)})$, respectively. We define

$$\phi_i^{(\alpha)} : \Omega^\bullet_{\mathfrak{X}_\alpha/M|V_i^{(\alpha)}} \to \Omega^\bullet_{\mathfrak{X}_\alpha|V_i^{(\alpha)}}$$

by

$$\begin{aligned}\phi_i^{(\alpha)}([\sum_{j_1<\cdots<j_p} a^{(\alpha)}_{j_1\cdots j_p}(x,t)dx^{(\alpha)}_{ij_1} \wedge \cdots \wedge dx^{(\alpha)}_{ij_p}]) \\ = \sum_{j_1<\cdots<j_p} a^{(\alpha)}_{j_1\cdots j_p}(x,t)dx^{(\alpha)}_{ij_1} \wedge \cdots \wedge dx^{(\alpha)}_{ij_p},\end{aligned}$$

where $[\sum_{j_1<\cdots<j_p} a^{(\alpha)}_{j_1\cdots j_p}(x,t)dx^{(\alpha)}_{ij_1} \wedge \cdots dx^{(\alpha)}_{ij_p}]$ is a local cross-section of the sheaf $\Omega^\bullet_{\mathfrak{X}_\alpha/M}$ over an open subset V_α^i, represented by a differential form

$$\sum_{j_1<\cdots<j_p} a^{(\alpha)}_{j_1\cdots j_p}(x,t)dx^{(\alpha)}_{ij_1} \wedge \cdots dx^{(\alpha)}_{ij_p}$$

involving $dx_{i1}^{(\alpha)}, \cdots, dx_{in_\alpha}^{(\alpha)}$ only. In what follows the proofs of Lemma 2.12 through Lemma 2.15 are straightfoward caluculations, so they will be omitted.

2.12 Lemma. $\phi_i^{(\alpha)}$ *splits the exact sequence* $\mathcal{O}_{V_i^{(\alpha)}}$*-modules*

$$0 \to G^1(\Omega^\bullet_{\mathfrak{X}_\alpha})_{|V_i^{(\alpha)}} \to \Omega^\bullet_{\mathfrak{X}_\alpha|V_i^{(\alpha)}} \to \Omega^\bullet_{\mathfrak{X}_\alpha/M|V_i^{(\alpha)}} \to 0$$

and satisfies

$$\phi_i^{(\alpha)} \cdot d_{\mathfrak{X}_\alpha/M} = d_{U_i^{(\alpha)}} \cdot \phi_i^{(\alpha)},$$

where $d_{\mathfrak{X}_\alpha/M}$ *is the differential of the complex* $\Omega^\bullet_{\mathfrak{X}_\alpha/M}$, *i.e., the relative exterior differentiation.*

Hereafter we use the notation $\omega(\alpha; p; i_0 \cdots i_q)$ ($\alpha \in \mathrm{Ob}(\square_n)$, p: a positive integer, and $i_0, \cdots, i_q \in \Lambda_\alpha$), which represents the component of $\beta \in (s(\mathrm{tot}\, \mathfrak{C}^\bullet(\mathcal{V}_\bullet, \mathcal{K}^\bullet_\bullet)))^r$ ($r = |\alpha| + p + q$) lying in $\Gamma(V_{i_0}^{(\alpha)} \cap \cdots \cap V_{i_q}^{(\alpha)}, \mathcal{K}^p_\alpha)$, for a complex of abelian sheaves $\mathcal{K}^\bullet_\bullet$ on $\mathfrak{X}_\bullet$. We define

$$\phi : K^\bullet(\Omega^\bullet_{\mathfrak{X}_\bullet/M}) \to K^\bullet(\Omega^\bullet_{\mathfrak{X}_\bullet})$$

by

$$(\phi\omega)(\alpha; p; i_0 \cdots i_q) := \phi_{i_0}^{(\alpha)}(\omega(\alpha; p; i_0 \cdots i_q))$$

for $\omega \in K^\bullet(\Omega^\bullet_{\mathfrak{X}_\bullet/M})$.

2.13 Lemma. *ϕ splits the exact sequence of abelian groups*

$$0 \to K^\bullet(G^1(\Omega^\bullet_{\mathfrak{X}_\bullet})) \to K^\bullet(\Omega^\bullet_{\mathfrak{X}_\bullet}) \to K^\bullet(\Omega^\bullet_{\mathfrak{X}_\bullet/M}) \to 0.$$

Define $J : K^\bullet(\Omega^\bullet_{\mathfrak{X}_\bullet M}) \to K^{\bullet+1}(\Omega^\bullet_{\mathfrak{X}_\bullet})$ by

$$(J\omega)(\alpha;p;i_0\cdots i_q) := (-1)^{p+|\alpha|+1}(\phi^{(\alpha)}_{i_0} - \phi^{(\alpha)}_{i_1})(\omega(\alpha;p;i_1\cdots i_q))$$

if $V^{(\alpha)}_{i_0} \cap V^{(\alpha)}_{i_1} \cap \cdots V^{(\alpha)}_{i_q} \neq \emptyset$. Then we have $J(K^\bullet(\Omega^\bullet_{\mathfrak{X}_\bullet/M})) \subset K^\bullet(G^1(\Omega^\bullet_{\mathfrak{X}_\bullet}))$.

2.14 Lemma. $\delta\phi - \phi\delta = J$

,where δ is the Cěch coboundary map.

Define the *total Lie derivative* $L_M : K^\bullet(\Omega^\bullet_{\mathfrak{X}_\bullet}) \to K^{\bullet+1}(\Omega^\bullet_{\mathfrak{X}_\bullet})$ with respect to the parameters of M by

$$(L_M\omega)(\alpha;p;i_0\cdots i_q) := (-1)^{|\alpha|}d_M(\omega(\alpha;p;i_0\cdots i_q))$$

Notice that $L_M(K^\bullet(G^i(\Omega^\bullet_{\mathfrak{X}_\bullet})) \subset K^\bullet(G^{i+1}(\Omega^\bullet_{\mathfrak{X}_\bullet}))$. We denote by

$$d_{\mathfrak{X}_\bullet} : K^\bullet(\Omega^\bullet_{\mathfrak{X}_\bullet}) \to K^{\bullet+1}(\Omega^\bullet_{\mathfrak{X}_\bullet})$$

the morphism of $\mathbb{C}$-vector spaces induced by the exterior differentiations $d_{\mathfrak{X}_\alpha} : \Omega^\bullet_{\mathfrak{X}_\alpha} \to \Omega^\bullet_{\mathfrak{X}_\alpha}$, and by

$$d_{\mathfrak{X}_\bullet/M} : K^\bullet(\Omega^\bullet_{\mathfrak{X}_\bullet/M}) \to K^{\bullet+1}(\Omega^\bullet_{\mathfrak{X}_\bullet/M})$$

the one induced by the relative exterior differentiations $d_{\mathfrak{X}_\alpha/M} : \Omega^\bullet_{\mathfrak{X}_\alpha/M} \to \Omega^{\bullet+1}_{\mathfrak{X}_\alpha/M}$. Combining Lemma 2.12 and (2.13), we obtain

2.15 Lemma.

$$(-1)^{|\alpha|}d_{\mathfrak{X}_\alpha}\phi(\omega(\alpha;p;i_0\cdots i_q)) = (L_M\phi + (-1)^{|\alpha|}\phi\, d_{\mathfrak{X}_\alpha/M})(\omega(\alpha;p;i_0\cdots i_q))$$

for $\omega(\alpha;p;i_0\cdots i_q) \in \Gamma(V^{(\alpha)}_{i_0} \cap \cdots \cap V^{(\alpha)}_{i_q}, \Omega^p_{\mathfrak{X}_\alpha/M})$.

We define the differential map $K^r(\Omega^\bullet_{\mathfrak{X}_\bullet}) \to K^{r+1}(\Omega^\bullet_{\mathfrak{X}_\bullet})$ of complexes of $\mathbb{C}$-vector spaces by

(2.14)

$$D^{(r)} := \oplus_{|\alpha|+p+q=r}\{\sum_{\substack{1\leq j\leq n+1\\ \alpha+e_j\in \mathrm{Ob}(\square_n)}} (-1)^{\varepsilon_j}d^{(p,q)*}_{\alpha,j} + (-1)^{|\alpha|}d^{(p,q)}_{\mathfrak{X}_\alpha} + (-1)^{|\alpha|}\delta^{(p,q)}_{\alpha}\}.$$

Here

$$d_{\alpha,j}^{(p,q)*} : \mathfrak{C}^q(\mathcal{V}_\alpha, \Omega^p_{\mathfrak{X}_\alpha}) \to \mathfrak{C}^q(\mathcal{V}_{\alpha+e_j}, \Omega^p_{\mathfrak{X}_{\alpha+e_j}}) \ \ (e_j = (0\cdots \overset{j}{1} \cdots 0) \in \mathbb{Z}^{n+1})$$

is the map induced by the holomorphic map $E_{\alpha\alpha+e_j} : \mathfrak{X}_{\alpha+e_j} \to \mathfrak{X}_\alpha$ over M corresponding to an arrow $\alpha \to \alpha + e_j$ in $\mathrm{Ob}(\Box_n)$,

$$\varepsilon_j = \alpha_0 + \alpha_1 + \cdots + \alpha_{j-2} \ \ (1 \le j \le n+1) \ \text{ for } \alpha = (\alpha_0 \cdots \alpha_n) \in \mathrm{Ob}(\Box_n),$$

where we understand $\varepsilon_1 = 0$,

$$d_{\mathfrak{X}_\alpha}^{(p,q)} : C^q(\mathcal{V}_\alpha, \Omega^p_{\mathfrak{X}_\alpha}) \to C^q(\mathcal{V}_\alpha, \Omega^{p+1}_{\mathfrak{X}_\alpha}) \ \text{ the exterior differentiation on } \mathfrak{X}_\alpha,$$

and

$$\delta_\alpha^{(p,q)} : C^q(\mathcal{U}_\alpha, \Omega^p_{\mathfrak{X}_\alpha}) \to C^{q+1}(\mathcal{U}_\alpha, \Omega^p_{\mathfrak{X}_\alpha}) \ \text{ the Cěch coboundary map.}$$

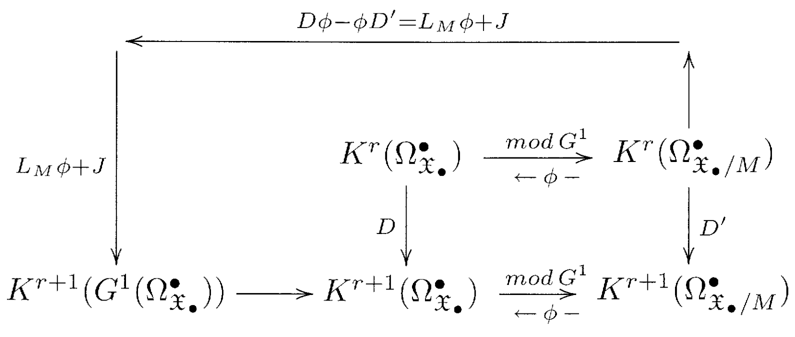
Similarly, we define the differential map $K^r(\Omega^\bullet_{\mathfrak{X}_\bullet/M}) \to K^{r+1}(\Omega^\bullet_{\mathfrak{X}_\bullet/M})$ by

(2.15)

$$D'^{(r)} := \oplus_{|\alpha|+p+q=r}\{ \sum_{\substack{1\le j\le n+1 \\ \alpha+e_j \in \Box_n}} (-1)^{\varepsilon_j} d_{\alpha,j}^{(p,q)\prime *} + (-1)^{|\alpha|} d_{\mathfrak{X}_\alpha/M}^{(p,q)} + (-1)^{|\alpha|}\delta_\alpha^{(p,q)\prime}\}.$$

Combining Lemma 2.14 and Lemma 2.15, we have the following:

Lemma 2.16.



Thus the connecting homomorphism associated to the exact sequence (2.11) *is induced by the morphism of* $\mathbb{C}$*-vector spaces*

$$L_M\phi + J : \ K^\bullet(\Omega^\bullet_{\mathfrak{X}_\bullet/M}) \to K^{\bullet+1}(G^1(\Omega^\bullet_{\mathfrak{X}_\bullet})).$$

Proof. We denote by D and D' the differentials of the single complexes of $\mathbb{C}$-vector spaces $K^\bullet(\Omega^\bullet_{\mathfrak{X}_\bullet})$ and $K^\bullet(\Omega^\bullet_{\mathfrak{X}_\bullet/M})$, defined in (2.14) and (2.15), respectively. Since

$$D = \sum_{\substack{1\le j\le n+1 \\ \alpha+e_j\in \mathrm{Ob}(\square^n)}} (-1)^{\varepsilon_j} d^*_{\alpha,j} + (-1)^{|\alpha|} d_{\mathfrak{X}_\alpha} + (-1)^{|\alpha|}\delta_\alpha \quad \text{on} \quad \mathfrak{C}^\bullet(\mathcal{V}_\alpha, \Omega^\bullet_{\mathfrak{X}_\alpha}) \text{ and,}$$

$$D' = \sum_{\substack{1\le j\le n+1 \\ \alpha+e_j\in \mathrm{Ob}(\square^n)}} (-1)^{\varepsilon_j} d^*_{\alpha,j} + (-1)^{|\alpha|} d_{\mathfrak{X}_\alpha/M} + (-1)^{|\alpha|}\delta_\alpha \quad \text{on} \quad \mathfrak{C}^\bullet(\mathcal{V}_\alpha, \Omega^\bullet_{\mathfrak{X}_\alpha/M}),$$

we have

$$\begin{aligned} D\phi - \phi D' &= \Big(\sum_{\substack{1\le j\le n+1 \\ \alpha+e_j\in \mathrm{Ob}(\square^n)}} (-1)^{\varepsilon_j} d^*_{\alpha,j}\Big)\phi - \phi\Big(\sum_{\substack{1\le j\le n+1 \\ \alpha+e_j\in \mathrm{Ob}(\square^n)}} (-1)^{\varepsilon_j} d^*_{\alpha,j}\Big) \\ &\quad + (-1)^{|\alpha|}(d_{\mathfrak{X}_\alpha}\phi - \phi d_{\mathfrak{X}_\alpha/M}) + (-1)^{|\alpha|}(\delta_\alpha\phi - \phi\delta_\alpha) \end{aligned} \tag{2.16}$$

on $\mathfrak{C}^\bullet(\mathcal{U}_\alpha, \Omega^\bullet_{\mathfrak{X}_\alpha/M})$. Since the map $E_{\alpha\alpha+e_j}$ $(1 \le j \le n+1)$ satisfies the requirement (iii) in (2.7), we have

$$\Big(\sum_{\substack{1\le j\le n+1 \\ \alpha+e_j\in \mathrm{Ob}(\square^n)}} (-1)^{\varepsilon_j} d^*_{\alpha,j}\Big)\phi - \phi\Big(\sum_{\substack{1\le j\le n+1 \\ \alpha+e_j\in \mathrm{Ob}(\square^n)}} (-1)^{\varepsilon_j} d^*_{\alpha,j}\Big) = 0.$$

Therefore Lemma 2.16 follows from (2.16), Lemma 2.14 and Lemma 2.15.

Q.E.D.

On each $V_i^{(\alpha)}$ we define the *total interior product* with respect to the parameters of M

$$I^i_\alpha : \Omega^\bullet_{\mathfrak{X}_\alpha|V_i^{(\alpha)}} \longrightarrow \Omega^\bullet_{\mathfrak{X}_\alpha|V_i^{(\alpha)}}$$

by

$$\begin{aligned} &I^i_\alpha\Big(\sum_{\substack{j_1<\cdots<j_r \\ k_1<\cdots<k_r \\ r+s=p \\ 0\le r,s}} a_{j_1\cdots j_r k_1\cdots k_s}(x,t)\,dx^{(\alpha)}_{ij_1}\wedge\cdots\wedge dx^{(\alpha)}_{ij_r}\wedge dt_{k_1}\wedge\cdots\wedge dt_{k_s}\Big) \\ &= \sum_{\substack{j_1<\cdots<j_r \\ k_1<\cdots<k_r \\ r+s=p \\ 0\le r,s}} s\cdot a_{j_1\cdots j_r k_1\cdots k_s}(x,t)\,dx^{(\alpha)}_{ij_1}\wedge\cdots\wedge dx^{(\alpha)}_{ij_r}\wedge dt_{k_1}\wedge\cdots\wedge dt_{k_s}\Big) \end{aligned} \tag{2.17}$$

for a local holomorphic p-form on $V_i^{(\alpha)}$. Here we understand the forms $dx^{(\alpha)}_{ij_1}\wedge\cdots\wedge dx^{(\alpha)}_{ij_r}\wedge dt_{k_1}\wedge\cdots\wedge dt_{k_s}$ for $r=0$ (resp. $s=0$) are those which do not involve $dx^{(\alpha)}_{i1},\cdots,dx^{(\alpha)}_{in_\alpha}$ (resp. $dt_1,\cdots,dt_m$). When $p=0$, we define $I^i_\alpha\equiv 0$. Notice that

$$I^i_\alpha(\Omega^\bullet_{\mathfrak{X}_\alpha|V_i^{(\alpha)}}) \subset G^1(\Omega^\bullet_{\mathfrak{X}_\alpha})_{|V_i^{(\alpha)}}.$$

Define

$$\lambda : K^\bullet(\Omega^\bullet_{\mathfrak{X}_\bullet}) \to K^{\bullet+1}(\Omega^\bullet_{\mathfrak{X}_\bullet})$$

by

$$(\lambda\omega)(\alpha;p;i_0\cdots i_q) := (-1)^{p+|\alpha|}(I^{i_0}_\alpha - I^{i_1}_\alpha)(\omega(\alpha;p;i_1\cdots i_q))$$

Lemma 2.17.

$$\lambda\phi \equiv J \ mod \ K^\bullet(G^2(\Omega^\bullet_{\mathfrak{X}_\bullet}))$$

Proof. It suffices to show that we have

$$\lambda(\phi\omega)(\alpha;p;i_0i_1\cdots i_p) - J\omega(\alpha;p;i_0i_1\cdots i_p) \in G^2(\Omega^p_{\mathfrak{X}_\alpha}). \tag{2.18}$$

for $\omega(\alpha;p;i_1\cdots i_q) \in \Gamma(V^{(\alpha)}_{i_1} \cap \cdots \cap V^{(\alpha)}_{i_q},\ \Omega^p_{\mathfrak{X}_\alpha/M})$ of the form

$$\omega(\alpha;p;i_1\cdots i_q) = [\mu(x,t)dx^{(\alpha)}_{i_1j_1} \wedge \cdots \wedge dx^{(\alpha)}_{i_1j_p}],$$

where $\mu(x,t)$ is a local holomorphic function on $V^{(\alpha)}_{i_1} \cap \cdots \cap V^{(\alpha)}_{i_q}$, and $[\mu(x,t)dx^{(\alpha)}_{i_1j_1} \wedge \cdots \wedge dx^{(\alpha)}_{i_1j_p}]$ a local cross-section of the sheaf $\Omega^p_{\mathfrak{X}_\alpha/M}$ over $V^{(\alpha)}_{i_1} \cap \cdots \cap V_{i_q}$ represented by the form $\mu(x,t)dx^{(\alpha)}_{i_1j_1} \wedge \cdots \wedge dx^{(\alpha)}_{i_1j_p}$. Indeed, since

$$\begin{aligned}
&\mu(x,t)dx^{(\alpha)}_{i_1j_1} \wedge \cdots \wedge dx^{(\alpha)}_{i_1j_p} \\
&\equiv \mu(x,t)\{\textstyle\sum_{j'_1<\cdots<j'_p} \frac{\partial(x^{(\alpha)}_{i_1j_1},\cdots,x^{(\alpha)}_{i_1j_p})}{\partial(x^{(\alpha)}_{i_0j'_1},\cdots,x^{(\alpha)}_{i_0j'_p})} dx^{(\alpha)}_{i_0j'_1} \wedge \cdots \wedge dx^{(\alpha)}_{i_0j'_p} \\
&\quad + \textstyle\sum_{k=1}^m \sum_{j'_1<\cdots<j'_{p-1}} \frac{\partial(x^{(\alpha)}_{i_1j_1},\cdots,x^{(\alpha)}_{i_1j_p})}{\partial(x^{(\alpha)}_{i_0j'_1},\cdots,x^{(\alpha)}_{i_0j'_{p-1}}\ t_k)} dx^{(\alpha)}_{i_0j'_1} \wedge\cdots\wedge dx^{(\alpha)}_{i_0j'_{p-1}} \wedge dt_k\} \\
&mod \quad G^2(\Omega^p_{\mathfrak{X}_\alpha}),
\end{aligned}$$

we have

$$\begin{aligned}
J\omega(\alpha;p;i_0i_1\cdots i_p) &= (-1)^{p+|\alpha|+1}(\phi^{(\alpha)}_{i_0} - \phi^{(\alpha)}_{i_1})\omega(\alpha;p;i_1\cdots i_q) \equiv \\
(-1)^{p+|\alpha|+1}\mu(x,t)\{-\textstyle\sum_{k=1}^m \sum_{j'_1<\cdots<j'_{p-1}} &\frac{\partial(x^{(\alpha)}_{i_1j_1},\cdots,x^{(\alpha)}_{i_1j_p})}{\partial(x^{(\alpha)}_{i_0j'_1},\cdots,x^{(\alpha)}_{i_0j'_{p-1}}\ t_k)} dx^{(\alpha)}_{i_0j'_1} \wedge \cdots \wedge dx^{(\alpha)}_{i_0j'_{p-1}} \wedge dt_k\} \\
&mod \quad G^2(\Omega^p_{\mathfrak{X}_\alpha})
\end{aligned} \tag{2.19}$$

On the other hand,

$$\lambda(\phi\omega)(\alpha;p;i_0i_1\cdots i_p)$$

$$= (-1)^{p+|\alpha|}(I_\alpha^{i_0} - I_\alpha^{i_1})(\mu(x,t)dx^{(\alpha)}_{i_1j_1} \wedge \cdots \wedge dx^{(\alpha)}_{i_1j_p})$$

$$(2.20) \qquad = (-1)^{p+|\alpha|}I_\alpha^{i_0}\{\mu(x,t)(\textstyle\sum_{j'_1<\cdots<j'_p} \frac{\partial(x^{(\alpha)}_{i_1j_1},\cdots,x^{(\alpha)}_{i_1j_p})}{\partial(x^{(\alpha)}_{i_0j'_1},\cdots,x^{(\alpha)}_{i_0j'_p})} dx^{(\alpha)}_{i_0j'_1} \wedge \cdots \wedge dx^{(\alpha)}_{i_0j'_p}$$

$$+ \textstyle\sum_{k=1}^m \sum_{j'_1<\cdots<j'_{p-1}} \frac{\partial(x^{(\alpha)}_{i_1j_1},\cdots,x^{(\alpha)}_{i_1j_p})}{\partial(x^{(\alpha)}_{i_0j'_1},\cdots,x^{(\alpha)}_{i_0j'_{p-1}}\ t_k)} dx^{(\alpha)}_{i_0j'_1} \wedge \cdots \wedge dx^{(\alpha)}_{i_0j'_{p-1}} \wedge dt_k) + \cdots)\} \equiv$$

$$(-1)^{p+|\alpha|}\mu(x,t)\{\textstyle\sum_{k=1}^m \sum_{j'_1<\cdots<j'_{p-1}} \frac{\partial(x^{(\alpha)}_{i_1j_1},\cdots,x^{(\alpha)}_{i_1j_p})}{\partial(x^{(\alpha)}_{i_0j'_1},\cdots,x^{(\alpha)}_{i_0j'_{p-1}}\ t_k)} dx^{(\alpha)}_{i_0j'_1} \wedge\cdots\wedge dx^{(\alpha)}_{i_0j'_{p-1}} \wedge dt_k\}$$

$$\mod \quad G^2(\Omega^p_{\mathfrak{X}_\alpha})$$

Then (2.18) follows from (2.19) and (2.20).

Q.E.D.

By Lemma 2.16 and Lemma 2.17 we infer that the connecting homomorphism associated to the exact sequence in (2.12) is induced from

$$K^\bullet(\Omega^\bullet_{\mathfrak{X}_\bullet/M}) \xrightarrow{\phi} K^\bullet(\Omega^\bullet_{\mathfrak{X}_\bullet}) \xrightarrow{L_M+\lambda} K^{\bullet+1}(G^1(\Omega^\bullet_{\mathfrak{X}_\bullet})) \xrightarrow{mod\, G^2} K^{\bullet+1}(Gr^1(\Omega^\bullet_{\mathfrak{X}_\bullet})).$$

Furthermore, since

$$K^\bullet(\Omega^\bullet_{\mathfrak{X}_\bullet/M}) \xrightarrow{\phi} K^\bullet(\Omega^\bullet_{\mathfrak{X}_\bullet})/K^\bullet(G^1(\Omega^\bullet_{\mathfrak{X}_\bullet})) \equiv K^\bullet(\Omega^\bullet_{\mathfrak{X}_\bullet/M})$$

is the identity map and

$$(L_M + \lambda)(K^\bullet(G^1(\Omega^\bullet_{\mathfrak{X}_\bullet})) \subset K^{\bullet+1}(G^2(\Omega^\bullet_{\mathfrak{X}_\bullet})),$$

we conclude that this connecting homomorphism is induced from $L_M + \lambda$: $K^\bullet(\Omega^\bullet_{\mathfrak{X}_\bullet}) \to K^{\bullet+1}(G^1(\Omega^\bullet_{\mathfrak{X}_\bullet}))$ by passing to quotients, i.e.,

$$(2.21) \qquad \begin{aligned} &K^\bullet(\Omega^\bullet_{\mathfrak{X}_\bullet/M}) = K^\bullet(\Omega^\bullet_{\mathfrak{X}_\bullet})/K^\bullet(G^1(\Omega^\bullet_{\mathfrak{X}_\bullet})) \\ &\xrightarrow{L_M+\lambda} K^{\bullet+1}(G^1(\Omega^\bullet_{\mathfrak{X}_\bullet}))/K^{\bullet+1}(G^2(\Omega^\bullet_{\mathfrak{X}_\bullet})) = K^{\bullet+1}(Gr^1(\Omega^\bullet_{\mathfrak{X}_\bullet})). \end{aligned}$$

Lemma 2.18.
$$(L_M + \lambda)D + D(L_M + \lambda) = 0$$

The proof of this lemma will be accomplished after proving several claims.

Claim 1.

$$d^*_{\alpha,j} L_M + L_M d^*_{\alpha,j} = 0 \quad \textit{and}$$
$$d^*_{\alpha,j}\lambda + \lambda d^*_{\alpha,j} = 0 \quad (1 \le j \le n+1)$$

Proof. Let $\omega \in K^r(\Omega^\bullet_{\mathfrak{X}_\bullet}) := \oplus_{|\alpha|+p+q=r} C^q(\mathcal{V}_\alpha, \Omega^p_{\mathfrak{X}_\alpha})$ and let $\beta \in \square_n$ be such that there exist $\alpha \in \square_n$ with $\beta = \alpha + e_j$, where $e_j = (0 \cdots \overset{\mathrm{j}}{1} \cdots 0)$. Then

$$(L_M d^*_{\alpha,j}\omega)(\beta; p; i_0 \cdots i_q) = (-1)^{|\beta|} d_M (E^*_{\alpha\beta}\omega(\alpha; p; \lambda_{\alpha\beta}(i_0), \cdots, \lambda_{\alpha\beta}(i_q)),$$

where $E_{\alpha\beta} : \mathfrak{X}_\beta \longrightarrow \mathfrak{X}_\alpha$ and $\lambda_{\alpha\beta} : \Lambda_\beta \longrightarrow \Lambda_\alpha$ are those defined in (2.7). On the other hand,

$$\begin{aligned}
&(d^*_{\alpha,j} L_M \omega)(\beta; p; i_0 \cdots i_q) \\
&= E^*_{\alpha\beta}(L_M\omega)(\alpha; p; \lambda_{\alpha\beta}(i_0), \cdots, \lambda_{\alpha\beta}(i_q)) \\
&= E^*_{\alpha\beta}((-1)^{|\alpha|} d_M \omega(\alpha; p; \lambda_{\alpha\beta}(i_0), \cdots, \lambda_{\alpha\beta}(i_q))
\end{aligned}$$

Since $E_{\alpha\beta} = e_{\alpha\beta} \times \mathrm{id}_M$ on $V^{(\beta)}_{i_0} \cap \cdots \cap V^{(\beta)}_{i_q}$, $d_M E^*_{\alpha\beta} = E^*_{\alpha\beta} d_M$. Hence

$$(-1)^{|\beta|}(L_M d^*_j \omega)(\beta; p; i_0 \cdots i_q) = (-1)^{|\alpha|}(d^*_j L_M \omega)(\beta; p; i_0 \cdots i_q),$$

which means $d^*_{\alpha,j} L_M + L_M d^*_{\alpha,j} = 0$ as $| \beta |=| \alpha | +1$. Similarly, we can show that $d^*_{\alpha,j}\lambda + \lambda d^*_{\alpha,j} = 0$.

Q.E.D.

Claim 2.

$$\delta_\alpha L_M + L_M \delta_\alpha = 0 \quad \textit{and} \quad \delta_\alpha \lambda + \lambda \delta_\alpha = 0$$

Proof. The first identity is trivial. We are going to show the second identity. Let $\omega \in \mathfrak{C}^{q-1}(\mathcal{V}_\alpha, \Omega^p_{\mathfrak{X}_\alpha})$. Then $\lambda\delta\omega \in \mathfrak{C}^{q+1}(\mathcal{V}_\alpha, \Omega^p_\alpha)$ is given by

$$(-1)^{|\alpha|}(\lambda\delta_\alpha\omega)(\alpha; p; i_0 \cdots i_{q+1}) = (-1)^p (I^{i_0}_\alpha - I^{i_1}_\alpha)(\delta_\alpha\omega)(\alpha; p; i_1 \cdots i_{q+1}) \tag{2.22}$$

$$= \sum_{j=1}^{q+1}(-1)^{j-1}(I_\alpha^{i_0} - I_\alpha^{i_1})\omega(\alpha; p; i_1 \cdots \check{i}_j \cdots i_{q+1}).$$

On the other hand,

$$(-1)^{|\alpha|}(\delta\lambda\omega)(\alpha; p; i_0 \cdots i_{q+1}) = (-1)^{p+|\alpha|}\sum_{j=0}^{q+1}(-1)^j(\lambda\omega)(\alpha; p; i_0 \cdots \check{i}_j \cdots i_{q+1})$$

$$\begin{aligned} &= (I_\alpha^{i_1} - I_\alpha^{i_2})\omega(\alpha; p; i_2 \cdots i_{q+1}) - (I_\alpha^{i_0} - I_\alpha^{i_2})\omega(\alpha; p; i_2 \cdots i_{q+1}) \\ &\quad + \sum_{j=2}^{q+1}(-1)^j(I_\alpha^{i_0} - I_\alpha^{i_1})\omega)(\alpha; p; i_1 \cdots \check{i}_j \cdots i_{q+1}). \end{aligned} \tag{2.23}$$

$$= -\{\sum_{j=1}^{q+1}(-1)^{j-1}(I_\alpha^{i_0} - I_\alpha^{i_1})\omega)(\alpha; p; i_1 \cdots \check{i}_j \cdots i_{q+1})\}.$$

From (2.22) and (2.23) it follows that

$$\{(\lambda\delta_\alpha + \delta_\alpha\lambda)\omega\}(\alpha; p; i_0 \cdots i_{q+1}) = 0.$$

Hence $\lambda\delta_\alpha + \delta_\alpha\lambda = 0$ as required.

Q.E.D.

Claim 3.

$$L_M d_{\mathfrak{X}_\alpha} + d_{\mathfrak{X}_\alpha} L_M = 0$$

Proof. Let $\omega \in \mathfrak{C}^q(\mathcal{V}_\alpha, \Omega^{p-2}_{\mathfrak{X}_\alpha})$. Then $L_M d_{\mathfrak{X}}.\omega \in \mathfrak{C}^q(\mathcal{V}_\alpha, \Omega^p_{\mathfrak{X}_\alpha})$ is given by

$$\begin{aligned} &(L_M d_{\mathfrak{X}}.\omega)(\alpha; p; i_0 \cdots i_q) \\ &= (-1)^{|\alpha|} d_M(d_{\mathfrak{X}}.\omega)(\alpha; \text{p}-1; i_0 \cdots i_q)) \\ &= (-1)^{|\alpha|+1} d_{\mathfrak{X}}.(d_M\omega)(\alpha; \text{p}-1; i_0 \cdots i_q) \\ &= (-1) d_{\mathfrak{X}}.(L_M\omega)(\alpha; \text{p-1}; i_0 \cdots i_q) \\ &= -(d_{\mathfrak{X}}.L_M\omega)(\alpha; p; i_0 \cdots i_q) \end{aligned}$$

Hence we have done.

Q.E.D.

Claim 4. *On each* $V_i^{(\alpha)}$ $(\alpha \in \mathrm{Ob}(\Box_n), i \in \Lambda_\alpha)$,

$$I_\alpha^i d_{\mathfrak{X}_\alpha} - d_{\mathfrak{X}_\alpha} I_\alpha^i = d_M.$$

Proof. It suffices to show that for a local holomorphic form ω on $V_i^{(\alpha)}$ of the form

$$\omega = \mu(x,t) dx_{ij_1}^{(\alpha)} \wedge \cdots \wedge dx_{ij_p}^{(\alpha)} \wedge dt_{k_1} \wedge \cdots dt_{k_q},$$

where $\mu(x,t)$ is a local holomorphic function,

$$I^i_\alpha d_{\mathfrak{X}_\alpha}\omega - d_{\mathfrak{X}_\alpha} I^i_\alpha \omega = d_M\omega$$

holds. We have,

$$d_{\mathfrak{X}_\alpha}\omega = \sum_{j\notin(j_1\cdots j_p)} \frac{\partial\mu(x,t)}{\partial x^{(\alpha)}_{ij}} dx^{(\alpha)}_{ij} \wedge dx^{(\alpha)}_{ij_1} \wedge \cdots \wedge dx^{(\alpha)}_{ij_p} \wedge dt_{k_1} \wedge \cdots dt_{k_q}$$

$$+ \sum_{j\notin(k_1\cdots k_q)} \frac{\partial\mu(x,t)}{\partial t_j} dt_j \wedge dx^{(\alpha)}_{ij_1} \wedge \cdots \wedge dx^{(\alpha)}_{ij_p} \wedge dt_{k_1} \wedge \cdots dt_{k_q}.$$

Hence

$$I^i_\alpha(d_{\mathfrak{X}_\alpha}\omega) = q(\sum_{j\notin(j_1\cdots j_p)} \frac{\partial\mu(x,t)}{\partial x^{(\alpha)}_{ij}} dx^{(\alpha)}_{ij} \wedge dx^{(\alpha)}_{ij_1} \wedge \cdots \wedge dx^{(\alpha)}_{ij_p} \wedge dt_{k_1} \wedge \cdots dt_{k_q})$$

(2.24)

$$+(q+1)(\sum_{j\notin(k_1\cdots k_q)} \frac{\partial\mu(x,t)}{\partial t_j} dt_j \wedge dx^{(\alpha)}_{ij_1} \wedge \cdots \wedge dx^{(\alpha)}_{ij_p} \wedge dt_{k_1} \wedge \cdots dt_{k_q})$$

On the other hand,

$$I^i_{(\alpha)}\omega = q\mu(x,t) dx^{(\alpha)}_{ij_1} \wedge \cdots \wedge dx^{(\alpha)}_{ij_p} \wedge dt_{k_1} \wedge \cdots dt_{k_q}).$$

Hence

$$d_{\mathfrak{X}_\alpha}(I^i_\alpha\omega) = q(\sum_{j\notin(j_1\cdots j_p} \frac{\partial\mu(x,t)}{\partial x^{(\alpha)}_{ij}} dx^{(\alpha)}_{ij} \wedge dx^{(\alpha)}_{ij_1} \wedge \cdots \wedge dx^{(\alpha)}_{ij_p} \wedge dt_{k_1} \wedge \cdots dt_{k_q})$$

(2.25)

$$+q(\sum_{j\notin(k_1\cdots k_p} \frac{\partial\mu(x,t)}{\partial t_j} dt_j \wedge dx^{(\alpha)}_{ij_1} \wedge \cdots \wedge dx^{(\alpha)}_{ij_p} \wedge dt_{k_1} \wedge \cdots dt_{k_q})$$

From (2.24) and (2.25) it follows that

$$I^i_\alpha(d_{\mathfrak{X}_\alpha}\omega) - d_{\mathfrak{X}_\alpha}(I^i_\alpha\omega) = d_M\omega.$$

Q.E.D.

Claim 5.

$$\lambda d_{\mathfrak{X}_\bullet} + d_{\mathfrak{X}_\bullet}\lambda = 0$$

Proof. Let $\omega \in \mathfrak{C}^{q-1}(\mathcal{V}_\alpha, \Omega^{p-1}_{\mathfrak{X}_\alpha})$. Then $\lambda d_{\mathfrak{X}.}\omega \in \mathfrak{C}^q(\mathcal{V}_\alpha, \Omega^p_{\mathfrak{X}_\alpha})$ is given by

$$\begin{aligned}
&(\lambda d_{\mathfrak{X}.}\omega)(\alpha; p; i_0\cdots i_q) \\
&= (-1)^{p+|\alpha|}(I^{i_0}_\alpha - I^{i_1}_\alpha)(d_{\mathfrak{X}_\alpha}\omega)(\alpha; p; i_0\cdots i_q) \\
&= (-1)^{p+|\alpha|} d_{\mathfrak{X}_\alpha}((I^{i_0}_\alpha - I^{i_1}_\alpha)\omega)(\alpha; p; i_0\cdots i_q) \quad (Claim 4) \\
&= -d_{\mathfrak{X}_\alpha}((-1)^{p-1+|\alpha|}(I^{i_0}_\alpha - I^{i_1}_\alpha)\omega))(\alpha; p; i_0\cdots i_q) \\
&= -d_{\mathfrak{X}_\alpha}(\lambda\omega)(\alpha; p; i_0\cdots i_q)
\end{aligned}$$

Hence

$$((\lambda d_{\mathfrak{X}.} + d_{\mathfrak{X}.}\lambda)\omega)(\alpha; p; i_0 \cdots i_q) = 0$$

This means $\lambda d_{\mathfrak{X}.} + d_{\mathfrak{X}.}\lambda = 0$.

Q.E.D.

Proof of Lemma 2.18. We are now ready to prove Lemma 2.18. By Claim 1, Claim 2, Claim 3 and Claim 5, we have

$$\begin{aligned}
&L_M D^{(r)} \\
&= L_M\{(\oplus_{\substack{|\alpha|+p+q\\=r}}\{\textstyle\sum_{1\le j\le n+1}(-1)^{\varepsilon_j} d_{\alpha,j}^{(p,q)*} + (-1)^{|\alpha|} d_{\mathfrak{X}_\alpha}^{(p,q)} + (-1)^{|\alpha|}\delta_\alpha^{(p,q)}\} \\
&= \{\oplus_{\substack{|\alpha|+p+q\\=r}}(\textstyle\sum_{1\le j\le n+1}(-1)^{\varepsilon_j} L_M d_{\alpha,j}^{(p,q)*} + (-1)^{|\alpha|} L_M d_{\mathfrak{X}_\alpha}^{(p,q)} \\
&\qquad\qquad\qquad\qquad + (-1)^{|\alpha|} L_M \delta_\alpha^{(p,q)})\} \\
&= (-1)\{(\oplus_{\substack{|\alpha|+p\\+q=r}}(\textstyle\sum_{1\le j\le n+1}(-1)^{\varepsilon_j} d_{\alpha,j}^{(p+1,q)*} L_M + (-1)^{|\alpha|} d_{\mathfrak{X}_\alpha}^{(p+1,q)} L_M \\
&\qquad\qquad\qquad\qquad + (-1)^{|\alpha|}\delta_\alpha^{(p+1,q)} L_M)\} \\
&= (-1)\{(\oplus_{\substack{|\alpha|+p+q\\=r+1}}(\textstyle\sum_{1\le j\le n+1}(-1)^{\varepsilon_j} d_{\alpha,j}^{(p,q)*} + (-1)^{|\alpha|} d_{\mathfrak{X}_\alpha}^{(p,q)} + (-1)^{|\alpha|}\delta_\alpha^{(p,q)}\} L_M \\
&= (-1) D^{(r+1)} L_M,
\end{aligned}$$

and similarly $\lambda D^{(r)} = (-1)D^{(r+1)}\lambda$. Therefore,

$$(L_M + \lambda)D^{(r)} = (-1)D^{(r+1)}(L_M + \lambda).$$

Q.E.D.

Consequently, by (2.12), (2.21) and Lemma 2.18, we conclude that the connection

$$\begin{aligned}
\nabla : \mathbb{H}^q(M, \pi_*(s(a_{1\bullet *}\Omega^\bullet_{\mathfrak{X}_\bullet/M})[1])) &\simeq H^q(K^\bullet(\Omega^\bullet_{\mathfrak{X}_\bullet/M})[1]) \\
&\to \Gamma(M, \Omega^1_M) \otimes_{\Gamma(M,\mathcal{O}_M)} \mathbb{H}^q(M, \pi_*(s(a_{1\bullet *}\Omega^\bullet_{\mathfrak{X}_\bullet/M})[1])) \\
&\simeq \Gamma(M, \Omega^1_M) \otimes_{\Gamma(M,\mathcal{O}_M)} H^q(K^\bullet(\Omega^\bullet_{\mathfrak{X}_\bullet/M})[1]) \\
&\simeq H^{q+1}(K^\bullet(Gr^1(\Omega^\bullet_{\mathfrak{X}_\bullet})[1])
\end{aligned}$$

is nothing but the homomorphism induced by $L_M + \lambda$ in (2.21). We should note that $L_M + \lambda$ is independent of the choice of a system of open coverings consisting of Stein coordinate neighborhoods of $\mathfrak{X}_\bullet$ subject to the conditions (i) through (iv) in (2.7) (cf. [2, p.220, (3.7.1)]).

Step(III) Proof of $\mathrm{Ker}\nabla = \mathrm{Im}\{R^\ell_{\mathbb{C}}(\pi) \to R^\ell_{\mathcal{O}}(\pi)\}$

Let $\mathcal{O}_\infty$ be the sheaf of germs of C^∞ functions on M. If $K^\bullet(\Omega^\bullet_{\infty\mathfrak{X}_\bullet/M}) := s(\oplus_{\alpha\in\square_n}\Gamma(\mathfrak{X}_\alpha, \Omega^\bullet_{\infty\mathfrak{X}_\alpha/M}))$ is a C^∞ analogue of $K^\bullet(\Omega^\bullet_{\mathfrak{X}_\bullet/M})$ constructed by use of the complex of relative C^∞ $\mathbb{C}$-valued differential forms on $\mathfrak{X}_\bullet$ over M, then we have locally

$$R^\ell_{\mathcal{O}_\infty}(\pi) := H^\ell(K^\bullet(\Omega^\bullet_{\infty\mathfrak{X}_\bullet/M})) \tag{2.26}$$

Furthermore, we can define the C^∞ analogue

$$\nabla_\infty : R^\ell_{\mathcal{O}_\infty}(\pi) \longrightarrow \Omega^1_M \otimes R^\ell_{\mathcal{O}_\infty}(\pi) \tag{2.27}$$

of the connection $\nabla : R^\ell_{\mathcal{O}}(\pi) \to \Omega^1_M \otimes R^\ell_{\mathcal{O}}(\pi)$ so that the following diagram commutes:

$$\begin{array}{ccccccc}
0 & \longrightarrow & R^\ell_{\mathbb{C}}(\pi) & \longrightarrow & R^\ell_{\mathcal{O}}(\pi) & \xrightarrow{\nabla} & \Omega^1_M \otimes R^\ell_{\mathcal{O}}(\pi) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & R^\ell_{\mathbb{C}}(\pi) & \longrightarrow & R^\ell_{\mathcal{O}_\infty}(\pi) & \xrightarrow{\nabla_\infty} & \Omega^1_{\infty M} \otimes R^\ell_{\mathcal{O}_\infty}(\pi)
\end{array}$$

Therefore it suffices to show that

$$Ker\,\nabla_\infty = Im\,\{R^\ell_{\mathbb{C}}(\pi) \to R^\ell_{\mathcal{O}_\infty}(\pi)\} \tag{2.28}$$

Since $\Omega^p_{\infty\mathfrak{X}_\alpha/M}$ $(0 \le p \le dim\,_{\mathbb{R}}\mathfrak{X}_\alpha, \alpha \in \square_n)$ are fine sheaves, the explicit calculation of ∇_∞ in terms of $H^\ell(K^\bullet(\Omega^\bullet_{\infty\mathfrak{X}_\bullet/M}))$ remain valid verbally for all coverings of $\mathfrak{X}_\bullet$ which are subject to the conditions (i) through (iv) in (2.7), except that they are Stein open coverings. Since, by [2, Proposition 2.5], the family $\pi_\cdot : \mathfrak{X}_\bullet \to M$ $(\pi_\bullet = \pi \circ a_\bullet)$ is C^∞ trivial at any point of M, we may take $\mathcal{V}_\alpha = \{\mathfrak{X}_\alpha\}$ for all $\alpha \in \square_n$ to calculate $H^\ell(K^\bullet(\Omega^\bullet_{\infty\mathfrak{X}_\bullet/M}))$ and

$$\nabla_\infty : H^\ell(K^\bullet(\Omega^\bullet_{\infty\mathfrak{X}_\bullet/M})) \to \Gamma(M, \Omega^1_{\infty M}) \otimes H^\ell(K^\bullet(\Omega^\bullet_{\infty\mathfrak{X}_\bullet/M})).$$

We fix these coverings. In what follows we shall use the following symbols:

$K^\bullet(\Omega^\bullet_{\infty\mathfrak{X}_\bullet}) := s(\oplus_{\alpha\in\square_n}\Gamma(\mathfrak{X}_\alpha, \Omega^\bullet_{\infty\mathfrak{X}_\alpha}))$,
$K^\bullet(G^p(\Omega^\bullet_{\infty\mathfrak{X}_\bullet})) := s(\oplus_{\alpha\in\square_n}\Gamma(\mathfrak{X}_\alpha, G^p(\Omega^\bullet_{\infty\mathfrak{X}_\alpha}))$ for $p \ge 1$,
etc..

The connection ∇_∞ in (2.27) is locally derived from

$$L_M + \lambda : K^\bullet(\Omega^\bullet_{\infty\mathfrak{X}_\bullet/M}) \to K^{\bullet+1}(F^1(\Omega^\bullet_{\infty\mathfrak{X}_\bullet/M}))$$

by passing to quotients, i.e.,

$$\begin{aligned} K^\bullet(\Omega^\bullet_{\infty\mathfrak{X}_\bullet/M}) &\cong K^\bullet(\Omega^\bullet_{\infty\mathfrak{X}_\bullet})/K^\bullet(G^1(\Omega^\bullet_{\infty\mathfrak{X}_\bullet})) \\ &\xrightarrow{L_M+\lambda} K^{\bullet+1}(G^1(\Omega^\bullet_{\infty\mathfrak{X}_\bullet}))/K^{\bullet+1}(G^2(\Omega^\bullet_{\infty\mathfrak{X}_\bullet})) \\ &\cong \Gamma(M,\Omega^1_{\infty M})\otimes_{\Gamma(M,\mathcal{O}_\infty)} K^\bullet(\Omega^\bullet_{\infty\mathfrak{X}_\bullet/M}) \end{aligned}$$

as in (2.21). Notice that λ is in fact zero map in this case, because we may take $\mathcal{V}_\alpha = \{\mathfrak{X}_\alpha\}$ for any $\alpha\in\square_n$. We have the C^∞ analogue of the exact sequence in (2.12):

$$(2.29)\quad \begin{aligned} &0 \to K^\bullet(G^1(\Omega^\bullet_{\infty\mathfrak{X}_\bullet}))/K^\bullet(G^2(\Omega^\bullet_{\infty\mathfrak{X}_\bullet})) \\ &\to K^\bullet(\Omega^\bullet_{\infty\mathfrak{X}_\bullet})/K^\bullet(G^2(\Omega^\bullet_{\infty\mathfrak{X}_\bullet})) \xrightarrow{P} K^\bullet(\Omega^\bullet_{\infty\mathfrak{X}_\bullet})/K^\bullet(G^1(\Omega^\bullet_{\infty\mathfrak{X}_\bullet})) \to 0. \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad K^\bullet(\Omega^\bullet_{\infty\mathfrak{X}_\bullet/M}) \end{aligned}$$

Notice that ∇_∞ comes from the connecting homomorphism of the long exact sequences of cohomology associated to this exact sequence. On the other hand, by Theorem 2.8, we have

$$\begin{aligned} R^\ell_{\mathbb{C}}(\pi) &\cong \mathbb{R}^\ell\pi_*(s(a_{1\bullet *}\mathbb{C}_{\mathfrak{X}_\bullet})[1]) \\ &\cong \mathbb{R}^\ell\pi_*(s(a_{1\bullet *}\Omega^\bullet_{\infty\mathfrak{X}_\bullet})[1]) \\ &\cong H^\ell(K^\bullet(\Omega^\bullet_{\infty\mathfrak{X}_\bullet})[1]) \quad \text{(locally)} \end{aligned}$$

Therefore, since the inclusion map $R^\ell_{\mathbb{C}}(\pi)\to R^\ell_{\mathcal{O}_\infty}(\pi)$ is induced from the composite of the projection

$$K^\bullet(\Omega^\bullet_{\infty\mathfrak{X}_\bullet}) \to K^\bullet(\Omega^\bullet_{\infty\mathfrak{X}_\bullet})/K^\bullet(G^2(\Omega^\bullet_{\infty\mathfrak{X}_\bullet}))$$

and the map P in (2.29), we infer that

$$(2.30)\qquad Im\,\{R^\ell_{\mathbb{C}}(\pi)\to R^\ell_{\mathcal{O}_\infty}(\pi)\}\subset Ker\,\nabla_\infty$$

Hence,

$$\begin{aligned} rank\,_{\mathcal{O}_\infty} R^\ell_{\mathcal{O}_\infty}(\pi) \geq rank\,_{\mathbb{C}}\, Ker\,\nabla_\infty &\geq rank\,_{\mathbb{C}}\, Im\,\{R^\ell_{\mathbb{C}}(\pi)\to R^\ell_{\mathcal{O}_\infty}(\pi)\} \\ &= rank\,_{\mathcal{O}_\infty} R^\ell_{\mathcal{O}_\infty}(\pi) \end{aligned}$$

From this (2.28) follows, and so we conclude that $\mathrm{Ker}\nabla = \mathrm{Im}\{R^\ell_{\mathbb{C}}(\pi)\to R^\ell_{\mathcal{O}}(\pi)\}$. This means that horizontal local cross-sections of $R^\ell_{\mathcal{O}}(\pi)$ with respect to ∇ coincides with local cross-sections of $R^\ell_{\mathbb{C}}$. That is, ∇ is the Gauss-Manin connection on $R^\ell_{\mathcal{O}}(\pi)$.

The fact $\nabla F^p(R^\ell_{\mathcal{O}}(\pi)) \subset \Omega^1_M \otimes F^{p-1}(R^\ell_{\mathcal{O}}(\pi))$ follows from the explicit calculation of ∇ in Step(II) (cf. (2.21)). This completes the proof of the assertion (iii) in Theorem 2.8.

§3 Infinitesimal mixed Torelli problem

In this section we shall formulate the infinitesimal mixed Torelli problem for an algebraic surface with ordinary singularities.

3.1 Definition. For a compact complex surface S with ordinary singularities, a complex analytic family of locally trivial deformations of S, parametrized by a complex manifold M, is defined to be a quintet $(\mathfrak{S}, \pi, M, o, \phi)$ such that;

(i) $(\mathfrak{S}, \pi, M)$ is a locally trivial analytic family of compact complex surfaces with ordinary singularities, parametrized by a complex space M (cf. Definition 2.1),

(ii) o is an assigned point of M,

(iii) $\phi : S \longrightarrow S_o := \pi^{-1}(o)$ is an isomorphism between compact complex spaces.

For a compact complex surface S with ordinary singularities, we put

$$\Theta_S := \mathcal{H}om_{\mathcal{O}_S}(\Omega^1_S, \mathcal{O}_S)$$

and call it the sheaf of germs of holomorphic vector fields on S. Now we are going to define the characteristic map

$$\sigma_o : T_oM \longrightarrow H^1(S, \Theta_S)$$

for an analytic family $(\mathfrak{S}, \pi, M, o, \phi)$ of locally trivial deformations of S, parametrized by a complex space M, where T_oM denotes the Zariski tengent space of M at o. In what follows, taking an open neighborhood of o in M, we may assume that M is a closed complex subspace of an open neighborhood B of the origin $0 = (0, \cdots, 0)$ of $\mathbb{C}^r$, and that the assigned point o coincides with the origin 0 of $\mathbb{C}^r$. We denote by $t = (t_1, \cdots, t_r)$ a system of local coordinates of $\mathbb{C}^r$.

3.2 Definition. A *geometrish Einspannung* of an analytic family $(\mathfrak{S}, \pi, M, o, \phi)$ of locally trivial deformations of a compact complex surface S with ordinary singularities, parametrized by a complex space M, a relative analogue of the *geometrish Einspannung* of a complex space in the sense of O. Forster and K. Knorr ([7]), is a data

$$\{\mathfrak{S}_i, \phi_i, \Phi_i, D_i, D_{ij}, u_{ij}, \mathfrak{T}_i, E_i, E_{ij}, g_{ij}\}_{i,j\in I},$$

such that;

(3.1)

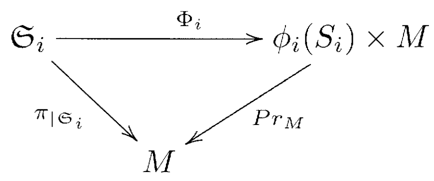
(i) $\{\mathfrak{S}_i\}_{i \in I}$ is a finite Stein covering of $\mathfrak{S}$,

(ii) D_i is an open polycyrinder in $\mathbb{C}^3$,

(iii) $\phi_i : S_i \longrightarrow D_i$ is a closed embedding, where $S_i := \mathfrak{S}_i \cap \pi^{-1}(o)$,

(iv) $\Phi_i : \mathfrak{S}_i \longrightarrow \phi_i(S_i) \times M \quad (\subset D_i \times M)$ is a biholomorphic map between complex spaces such that:

(a) the diagram



commuts,

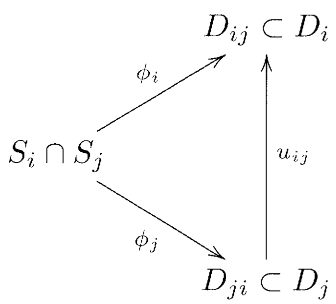
(b) $\Phi_{i|\mathfrak{S}_i} = \phi_i$,

(v) $D_{ij} \subset D_i$ is a Stein open subset,

(vi) $u_{ij} : D_{ji} \longrightarrow D_{ij}$ is a biholomorphic map between complex manifolds such that:

(a) $S_i \cap S_j = \phi_i^{-1}(D_{ij} \cap \phi_i(S_i))$,

(b) for any $i, j \in I$ with $S_i \cap S_j \neq \emptyset$, the diagram



commutes,

(c) for any $i, j, k \in I$ with $S_i \cap S_j \cap S_k \neq \emptyset$,

$$u_{ik} \equiv u_{ij} \circ u_{jk} \ \text{ modulo the ideal sheaf } \mathcal{I}_{\phi_k(S_k)} \text{ on } D_{ki} \cap D_{kj},$$

(d) for any $i \in I$, $D_{ii} = D_i$ and $u_{ii} = id_{D_i}$ (the identity on D_i),

(vii) $E_i \subset\subset D_i$ is a relatively compact open polycyrinder D_i such that if we put $T_i := \phi_i^{-1}(E_i)$ and $E_{ij} := E_i \cap D_{ij}$, then

$$(T_i, \phi_{i|T_i}, E_i, E_{ij}, u_{ij|E_{ij}})$$

is a refinement of the system $(S_i, \phi_i, D_i, D_{ij}, u_{ij})$,

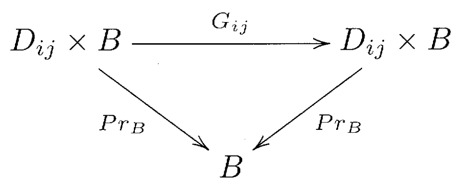
(viii) if we put $\mathfrak{T}_i := \Phi_i^{-1}(E_i \times M)$, then $\{\mathfrak{T}_i\}_{i \in I}$ is a finite Stein covering of $\mathfrak{S}$,

(ix) for any $i, j \in I$ with $D_{ij} \neq \emptyset$,

$$G_{ij} : D_{ij} \times B \longrightarrow D_{ij} \times B$$

is a biholomorphic map between complex manifolds such that:

(a) the diagram



commuts, i.e., G_{ij} is a vertical automorphism over B,

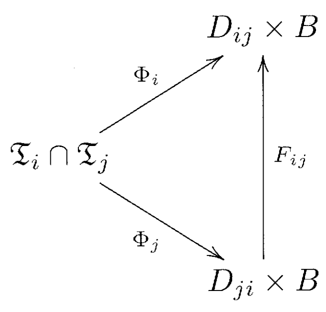
(b) $G_{ij|D_{ij} \times 0} = id_{E_{ij} \times 0}$ (the identity on $E_{ij} \times 0$),

(c) $G_{ii} = id_{D_i \times B}$,

(x) the automorphism G_{ij} send $\mathcal{I}((\phi_i(T_i) \cap E_{ij}) \times M)$, the ideal sheaf of $(\phi_i(T_i) \cap E_{ij}) \times M)$ in $\mathcal{O}_{E_{ij} \times B}$, into itself,

(xi) if we put $F_{ij} := G_{ij} \circ (u_{ij} \times id_B)$, then

(a) for any $i, j \in I$ with $\mathfrak{T}_i \cap \mathfrak{T}_j \neq \emptyset$, the diagram



commutes,

(b) for any $i, j, k \in I$ with $E_{ijk} := E_{ij} \cap E_{ik} \neq \emptyset$,

$$F_{ik} \equiv F_{ij} \circ F_{jk}$$

modulo the ideal sheaf $\mathcal{I}_{\phi_k(T_k) \times M}$ on $(E_{ki} \cap E_{kj}) \times B$.

Let (x_i^1, x_i^2, x_i^3) be a system of local coordinates on D_i. Since D_{ij} is an open subset of D_i, we may regard this as a system of local cordinates on D_{ij}. We denote by (x_i, t) a set of r+3 complex numbers $x_i^1, x_i^2, x_i^3, t_1, \cdots, t_r$ and also the point $D_i \times \mathbb{C}^r$ with the coordinates $(x_i^1, x_i^2, x_i^3, t_1, \cdots, t_r)$. We express the vertical automorphism $G_{ij} : D_{ij} \times B \longrightarrow D_{ij} \times B$ over B as

$$(x_i^1, x_i^2, x_i^3, t) = G_{ij}(x_i, t) = (g_{ij}^1(x_i, t), g_{ij}^2(x_i, t), g_{ij}^3(x_i, t), t)$$

and the holomorphic map $F_{ij} : D_{ji} \times B \longrightarrow D_{ij} \times B$ as

$$(x_i^1, x_i^2, x_i^3, t) = F_{ij}(x_j, t) = (f_{ij}^1(x_j, t), f_{ij}^2(x_j, t), f_{ij}^3(x_j, t), t).$$

Then, since $F_{ij} = G_{ij} \circ (u_{ij} \times id_B)$, we have

$$x_i^\alpha = f_{ij}^\alpha(x_j, t) = g_{ij}^\alpha(u_{ij}(x_j), t) \quad (1 \le \alpha \le 3) \tag{3.2}$$

We identify T_oM, the tangent space of the parameter space M at o, with the subspace

$$\{\ v \in T_o\mathbb{C}^r \ \mid (v\xi)(o) = 0 \text{ for any } \xi \in \mathcal{I}(M)_o\ \}$$

of $T_o\mathbb{C}$, where $\mathcal{I}(M)_o$ denotes the stalk at o of the sheaf ideal of M in $\mathcal{O}_{\mathbb{C}^r}$. For any $\partial/\partial t \in T_oM$, we set

$$\begin{aligned} \hat{\theta}_{ij} &:= \Sigma_{\alpha=1}^3 u_{ji}^*(\frac{\partial f_{ij}^\alpha}{\partial t}(x_j, o))(\frac{\partial}{\partial x_i^\alpha}) \\ &= \Sigma_{\alpha=1}^3 \frac{\partial g_{ij}^\alpha}{\partial t}(x_i, o)(\frac{\partial}{\partial x_i^\alpha}) \quad \text{on} \quad D_{ij} \subset D_i, \end{aligned} \tag{3.3}$$

where u_{ji}^* denotes the pull-back of holomorphic functions on D_{ji} by the map $u_{ji} : D_{ij} \longrightarrow D_{ji}$, and

$$\frac{\partial g_{ij}^\alpha}{\partial t}(x_i, 0) = \Sigma_{\lambda=1}^r v_\lambda \frac{\partial g_{ij}^\alpha}{\partial t_\lambda}(x_i, 0) \qquad \text{for} \quad \frac{\partial}{\partial t} = \Sigma_{\lambda=1}^r v_\lambda (\frac{\partial}{\partial t_\lambda})_o.$$

Claim 1. *$\hat{\theta}_{ij|E_{ij}} \in \Gamma(E_{ij}, \Theta_{E_i}(-\log\phi_i(T_i)))$, where $\Theta_{E_i}(-\log\phi_i(T_i)))$ denotes the sheaf of germs of logarithmic vector fields along $\phi_i(T_i)$ on E_i.*

Proof of Claim 1. Let $\xi_1(t), \cdots, \xi_s(t)$ be the generators of the ideal sheaf of M in $\mathcal{O}_B$, $h_i(x_i)$ the generators of the sheaf of $\phi_i(T_i)$ in $\mathcal{O}_{E_i}$. Then, by the condition (ix) in (3.1), there exist holomorphic functions $a_i(x_i,t)$, $b_i^\beta(x_i,t)$, $1 \le \beta \le s$, on $E_{ij} \times B$ such that

$$(3.4) \qquad h_i(g_{ij}(x_i,t)) = a_i(x_i,t)h_i(x_i) + \Sigma_{\beta=1}^s b_i^\beta(x_i,t)\xi_\alpha(t) \qquad \text{on} \quad E_{ij} \times B.$$

Derivating the both sides of (3.4) by $\partial/\partial t = \Sigma_{\lambda=1}^r v_\lambda(\partial/\partial t_\lambda)_o \in T_oM$, we have

$$(3.5) \qquad \begin{aligned} &\Sigma_{\alpha=1}^3 \frac{\partial g_{ij}^\alpha}{\partial t}(x_i,t)\frac{\partial h_i}{\partial x_i^\alpha}(g_{ij}(x_i,t)) \\ &= \frac{\partial a_i}{\partial t}(x_i,t)h_i(x_i) + \frac{\partial}{\partial t}(\Sigma_{\beta=1}^s b_1^\beta(x_i,t)\xi_\alpha(t)). \end{aligned}$$

Since $g_{ij}(x_i,o) = x_i$ and $(\partial\xi/\partial t)_{|t=0} = 0$ for any $\xi \in \mathcal{I}(M)_0$, substituting 0 for t in the equation (3.5), we have

$$\Sigma_{\alpha=1}^3 \frac{\partial g_{ij}^\alpha}{\partial t}(x_i,0)\frac{\partial h_i}{\partial x_i^r}(x_i) = \frac{\partial a_i}{\partial t}(x_i,0)h_i(x_i),$$

which indicates that the vector field $\hat{\theta}_{ij}$ sends the ideal sheaf of $\phi_i(T_i)$ into itself, that is, $\hat{\theta}_{ij|E_{ij}}$, the restriction of $\hat{\theta}_i j$ to E_{ij}, is a logarithmic tangent vector fields along $\phi_i(T_i)$ on E_{ij}.

Q.E.D. for **Claim 1**

Since

$$\Theta_{\phi_i(T_i)} = \Theta_{\phi_i(E_i)}(-\log\phi_i(T_i))/\Theta_{\phi_i(E_i)}(-\phi_i(T_i)),$$

where $\Theta_{\phi_i(E_i)}(-\phi_i(T_i))$ denotes the sheaf of germs of holomorphic tangent vector fields on E_i which vanish on $\phi_i(T_i)$, $\hat{\theta}_{ij|E_{ij}}$ determines an element of $\Gamma(E_{ij}, \Theta_{\phi_i(T_i)})$, which we denote by $theta_{ij}^*$, and by θ_{ij} the element of $\Gamma(T_i \cap T_j, \Theta_{S_0})$ corresponding to θ_{ij}^*, i.e.,

$$\theta_{ij} := (\phi_i^{-1})_*(\Sigma_{\alpha=1}^3 \frac{\partial g_{ij}^\alpha}{\partial t}(x_i,o)(\frac{\partial}{\partial x_i^\alpha})_{|\phi_i(T_i)\cap E_{ij}}).$$

Claim 2. *For any $i, j, k \in I$ with $T_i \cap T_j \cap T_k \neq \emptyset$, we have*

$$\theta_{ij} + \theta_{jk} + \theta_{ki} \equiv 0 \qquad on \quad T_i \cap T_j \cap T_k \tag{3.6}$$

Proof of Claim 2. From the condition (xi),(b) in (3.1), it follows that

(3.7) $x_i^\alpha = f_{ij}^\alpha(f_{jk}(x_k, t), t) = f_{ik}^\alpha(x_k, t)$
modulo the ideal sheaf $\mathcal{I}_{\phi_k(T_k)\times M}$ on $(E_{ki} \cap E_{kj}) \times B$
$(1 \le \alpha \le 3)$.

Since $f_{jk}(x_k, 0) = u_{jk}(x_k)$, derivating the both sides of (3.7) by $\partial/\partial t = \Sigma_{\lambda=1}^r v_\lambda(\partial/\partial t_\lambda)_o \in T_o M$ and substituting 0 for t, we have

(3.8)

$$\frac{\partial f_{ij}^\alpha}{\partial t}(u_{jk}(x_k), 0) + \Sigma_{\beta=1}^3 \frac{\partial f_{jk}^\beta}{\partial t}(x_k, 0) \cdot \frac{\partial u_{ij}^\alpha}{\partial x_j^\beta}(u_{jk}(x_k)) \equiv \frac{\partial f_{ik}^\alpha}{\partial t}(x_k, 0)$$

modulo the ideal sheaf $\mathcal{I}_{\phi_k(T_k)}$ on $E_{ki} \cap E_{kj}$

Then, since

$$u_{ki}^*(\frac{\partial f_{ij}^\alpha}{\partial t}(u_{jk}(x_k), 0)) = u_{ji}^*(\frac{\partial f_{ij}^\alpha}{\partial t}(x_j, 0)),$$

it follows from (3.8) that

(3.9)

$$\Sigma_{\alpha=1}^3 u_{ji}^*(\frac{\partial f_{ij}^\alpha}{\partial t}(x_j, 0))(\frac{\partial}{\partial x_i^\alpha}) + \Sigma_{\alpha=1}^3 u_{ki}^*(\Sigma_{\beta=1}^3 \frac{\partial f_{jk}^\beta}{\partial t}(x_k, 0)) \cdot \frac{\partial u_{ij}^\alpha}{\partial x_j^\beta}(u_{jk}(x_k))(\frac{\partial}{\partial x_i^\alpha})$$

$$\equiv \Sigma_{\alpha=1}^3 u_{ki}^*(\frac{\partial f_{ik}^\alpha}{\partial t}(x_k, 0))(\frac{\partial}{\partial x_i^\alpha})$$

modulo the ideal sheaf $\mathcal{I}_{\phi_i(T_i)}$ on $E_{ij} \cap E_{ik}$.

On the other hand,

$$(u_{ij})_* \hat{\theta}_{jk} = \Sigma_{\alpha=1}^3 (u_{ij})_* \{u_{kj}^*(\frac{\partial f_{jk}^\alpha}{\partial t}(x_k, 0))(\frac{\partial}{\partial x_j^\alpha})\}$$

$$= \Sigma_{\alpha=1}^3 u_{ji}^* u_{kj}^*(\frac{\partial f_{jk}^\alpha}{\partial t}(x_k, 0))\{\Sigma_{\beta=1}^3 u_{ji}^*(\frac{\partial u_{ij}^\beta}{\partial x_j^\alpha}(x_j))(\frac{\partial}{\partial x_i^\beta})\}$$

$$= \Sigma_{\beta=1}^3 \{\Sigma_{\alpha=1}^3 u_{ki}^*(\frac{\partial f_{jk}^\alpha}{\partial t}(x_k, 0)) \frac{\partial u_{ij}^\beta}{\partial x_j^\alpha}(u_{ji}(x_i))\}(\frac{\partial}{\partial x_i^\beta})$$

$$= \Sigma_{\beta=1}^{3} u_{ki}^{*}\{\Sigma_{\alpha=1}^{3} \frac{\partial f_{jk}^{\alpha}}{\partial t}(x_k, 0)) \frac{\partial u_{ij}^{\beta}}{\partial x_j^{\alpha}}(u_{jk}(x_k))\}(\frac{\partial}{\partial x_i^{\beta}})$$

Therefore, by (3.8), we have

$$(3.10) \qquad \hat{\theta}_{ij}^{*} + (u_{ij})_{*}\hat{\theta}_{jk}^{*} \equiv \hat{\theta}_{ik}^{*}$$

modulo the ideal sheaf $\mathcal{I}_{\phi_i(T_i)}$ on $E_{ij} \cap E_{ik}$.

Further, derivating the congruence equation

$$f_{ik}(f_{ki}(x_i, t), t) \equiv id_{D_i \times B}$$

modulo the ideal sheaf $\mathcal{I}_{\phi_i(T_i)\times M}$ on $E_i \times B$

by $\partial/\partial t = \Sigma_{\lambda=1}^{r} v_\lambda(\partial/\partial t_\lambda)_o \in T_oM$, and substituting 0 for t, we have

$$\frac{\partial f_{ik}^{\alpha}}{\partial t}(u_{ki}(x_i, 0) + \Sigma_{\beta=1}^{3} \frac{\partial f_{ki}^{\beta}}{\partial t}(x_i), 0)) \frac{\partial u_{ik}^{\alpha}}{\partial x_k^{\beta}}(u_{ki}(x_i)) \equiv 0$$

modulo the ideal sheaf $\mathcal{I}_{\phi_i(T_i)}$ on E_{ik}.

Therefore, we have

$$\hat{\theta}_{ik} := \Sigma_{\alpha=1}^{3}(u_{ki})^{*}(\frac{\partial f_{ik}^{\alpha}}{\partial t}(x_k, 0))(\frac{\partial}{\partial x_i^{\alpha}})$$

$$\equiv -\Sigma_{\alpha=1}^{3} u_{ki}^{*}\{\Sigma_{\beta=1}^{3} \frac{\partial f_{ki}^{\beta}}{\partial t}(u_{ik}(x_k), 0) \frac{\partial u_{ik}^{\alpha}}{\partial x_k^{\beta}}(x_k)\}(\frac{\partial}{\partial x_i^{\alpha}})$$

(3.11)

$$= -(u_{ik})_{*}\{\Sigma_{\alpha=1}^{3} u_{ik}^{*}(\frac{\partial f_{ki}^{\alpha}}{\partial t}(x_i, 0))(\frac{\partial}{\partial x_k^{\alpha}})\}$$

$$= -(u_{ik})_{*}\hat{\theta}_{ki}.$$

modulo the ideal sheaf $\mathcal{I}_{\phi_i(T_i)}$ on E_{ik}.

Consequently, by (3.10) and (3.11), we have

$$\theta_{ij}^{*} + (u_{ij})_{*}\theta_{jk}^{*} + (u_{ik})_{*}\theta_{ki}^{*} \equiv 0 \qquad \text{on} \quad \phi_i(T_i) \cap E_{ij} \cap E_{ik},$$

from which (3.6) follows.

Q.E.D. for **Claim 2**

By claim 2, we conclude that the collection $\{ \theta_{ij}\}_{i,j\in I}$ define an element of $H^1(S_0, \Theta_{S_0})$, which is independent of the choice of the system

$$\{\mathfrak{S}_i, \phi_i, \Phi_i, D_i, D_{ij}, u_{ij}, \mathfrak{T}_i, E_i, E_{ij}, g_{ij}\}_{i,j\in I}.$$

3.3 Definition. We define the characteristic map $\sigma_o : T_oM \longrightarrow H^1(S_o, \Theta_{S_o})$ of a complex analytic family $(\mathfrak{S}, \pi, M, o, \phi)$ of locally trivial deformations of S, parametrized by a complex space M, by $\sigma_o(\partial/\partial t) = \{\theta_{ij}\}_{i,j\in I}$ for any $\partial/\partial t \in T_oM$, where T_oM denotes the Zariski tangent space of M at o.

3.4 Definition. For a locally trivial complex analytic family $(\mathfrak{S}, \pi, M)$ of compact complex surfaces with ordinary singularities, if the characteristic map $\sigma_t : T_tM \longrightarrow H^1(S_t, \Theta_{S_t})$ is injective at $t \in M$, then we say that the family $(\mathfrak{S}, \pi, M)$ is *effective at* $t \in M$. If the family $(\mathfrak{S}, \pi, M)$ is effective at every $t \in M$, then we say that the family is *effectively parametrized.*

Now we proceed to defining the completeness. First, we recall the definition of a complex analytic family of deformations, not necessarily locally trivial, of a compact complex space, and define the completeness for such a family.

3.5 Definition. A complex analytic family of compact complex spaces, parametrized by a complex space, is a triplet $(\mathfrak{Y}, \varpi, N)$ such that $\varpi : \mathfrak{Y} \longrightarrow N$ is a proper, flat surjective holomorphic map between complex spaces.

3.6 Definition. For a compact complex space Y, a complex analytic family of deformatins of Y, parametrized by a complex space N, is defined to be a quintet $(\mathfrak{Y}, \varpi, N, o, \psi)$ such that;

(i) $(\mathfrak{Y}, \varpi, N)$ is a complex analytic family of compact complex spaces, parametrized by the complx space N,

(ii) o is an assigned point of N,

(iii) $\psi : Y \longrightarrow Y_o := \varpi^{-1}(o)$ is a biholomorphic map betwen complex spaces.

Suppose we are given a complex analytic family $(\mathfrak{Y}, \varpi, N)$ of compact complex spaces, parametrized by a complex space N. Let N' be another complex space, and $h : N' \longrightarrow N$ a holomorphic map from N' to N. We put $h^*\mathfrak{Y} := \mathfrak{Y} \times_N N'$, the fiber product of $\mathfrak{Y}$ and N' over N, and $Pr_{N'} : h^*\mathfrak{Y} \longrightarrow N'$, the projection to N'. Then $(h^*\mathfrak{Y}, Pr_{N'}, N')$ is a complex analytic family of complex spaces, parametrized by the complex space N'.

3.7 Definition. The family $(h^*\mathfrak{Y}, Pr_{N'}, N')$ thus obtained is called the family induced from $(\mathfrak{Y}, \varpi, N)$ by the holomorphic map $h : N' \longrightarrow N$. In particular, if N' is a complex subspace of N, and if $h : N' \longrightarrow N$ is the inclusion map, then we call the family $(h^*\mathfrak{Y}, Pr_{N'}, N')$ the restriction of $(\mathfrak{Y}, \varpi, N)$ to N' and denote it by $(\mathfrak{Y}_{|N'}, \varpi, N')$.

3.8 Definition. Let $(\mathfrak{Y}, \varpi, N, p, \psi)$ and $(\mathfrak{Y}', \varpi', N', p', \psi')$ be two complex analytic family of deformations of the same complex space Y, parametrized by complex spaces. If there is a holomorphic map $h : N' \longrightarrow N$ with $h(p') = p$ such that $\mathfrak{Y}'$ is isomorphic to $h^*\mathfrak{Y}$ i.e., there is a biholomorphic map $\Phi : h^*\mathfrak{Y} \longrightarrow \mathfrak{Y}^*$ over the identity map $id_{N'} : N' \longrightarrow N'$, then we say that the family $(\mathfrak{Y}', \varpi', N', p', \psi')$ is *induced* from the family $(\mathfrak{Y}, \varpi, N, p, \psi)$.

3.9 Definition. Let $(\mathfrak{Y}, \varpi, N)$ be a complex analytic family of compact complex spaces, parametrized by a complex space, and let p is a point of N. $(\mathfrak{Y}, \varpi, N)$ is said to be *complete at* $p \in N$ if for any complex analytic family $(\mathfrak{Y}', \varpi', N', p', \psi')$ of deformations of $S_p := \varpi^{-1}(p)$, parametrized by a complex space, there are a sufficiently small open neighborhood N'' of p' in N' such that the restriction $(\mathfrak{Y}'_{|N''}, \varpi', N'', p', \psi')$ of $(\mathfrak{Y}', \varpi', N', p', \psi')$ to N'' is induced from $(\mathfrak{Y}, \varpi, N, p, id_{S_p})$. Here $(\mathfrak{Y}, \varpi, N, p, id_{S_p})$ denotes the complex analytic family $(\mathfrak{Y}, \varpi, N)$ considerd as a complex analytic family of deformations of $S_p := \varpi^{-1}(p)$ with $id_{S_p} : S_p := \varpi^{-1}(p) \longrightarrow S_p$, the identity map on S_p.

We are now in a position to define the completeness with respect to locally trivial deformation for a locally trivial complex analytic family of compact complex surfaces with ordinary singulatities.

3.10 Definition. Let $(\mathfrak{Y}, \varpi, M)$ be a locally trivial complex analytic family of compact complex surfaces with ordinary singularities, parametrized by a complex space. $(\mathfrak{Y}, \varpi, N)$ is said to be *complete at* $t_0 \in M$ *with respect to locally trivial deformations* if for any complex analytic family $(\mathfrak{Y}', \varpi', M', o', \phi')$ of locally trivial deformations of $S_{t_o} := \varpi^{-1}(t_o)$, parametrized by a complex space, there are a sufficiently small open neighborhood N' of o' in M', and a holomorphic map $h : N' \longrightarrow M$ with $h(o') = t_o$ such that the restriction $(\mathfrak{Y}'_{|N}, \varpi', N', o', \phi')$ of $(\mathfrak{Y}', \varpi', N', o', \phi')$ to N is induced from $(\mathfrak{Y}, \varpi, M, t_o, id_{S_{t_o}})$.

3.11 Definition. The family $(\mathfrak{Y}, \varpi, M)$ is called *complete with respect to locally trivial deformations* if $(\mathfrak{Y}, \varpi, M)$ is complete at every point $t \in M$ with respect to locally trivial deformations.

3.12 Theorem. *For a compact complex surface S with ordinary singularities, there exists the Kuranishi family $(\mathfrak{Y}, \pi, M, o, \phi)$ with respect to locally trivial deformations of S, that is, the family for which the following conditions are satisfied:*

(i) *$(\mathfrak{Y}, \pi, N, o, \phi)$ is a complex analytic family of locally trivial deformations of S, parametrized by the complx space N,*

(ii) *the characteristic map at o*

$$\sigma_o : T_o M \longrightarrow H^1(S_o, \Theta_{S_o})$$

is injecticve,

(iii) *$(\mathfrak{Y}, \pi, M)$ is complete with respect to locally trivial deformations.*

Proof. Due to a result by H. Grauert ([10]), or V. P. Paramodov ([19]), or O. Forster and K. Knorr ([7]), there exists a complex analytic family $(\overline{\mathfrak{S}}, \overline{\pi}, \overline{M}, \overline{o}, \overline{\phi})$ of deformations of S, not necessarily locally trivial, parametrized by a complex space $\overline{M}$, such that;

(i) the characteristic map at $\overline{o}$

$$\tau_o : T_{\overline{o}}\overline{M} \longrightarrow \mathrm{Ext}_{\mathcal{O}_{S_{\overline{o}}}}(\Omega^1_{S_{\overline{o}}}, \mathcal{O}_{S_{\overline{o}}})$$

is defined, and is injective,

(ii) the complex analytic family $(\overline{\mathfrak{S}}, \overline{\pi}, \overline{M})$ of compact complex spaces is *complete* at every point $t \in \overline{M}$.

By Cororally (0.2) in [6], for any point $x \in S_{\overline{o}}$, there exists a locally closed complex subspace M_x of $\overline{M}$ containing the point $\overline{o}$, which enjoys the following property:

> If $\alpha : (N', o') \longrightarrow (\overline{M}, \overline{o})$ is a holomorphic map between germs of complex spaces, then the induced family $(\alpha^*\overline{\mathfrak{S}}, x) \longrightarrow (N', o')$ of deformations of the germ of complex space $(S_{\overline{o}}, x)$ is isomorphic to the trivial deformation $(S_{\overline{o}}, x) \times (N', o') \longrightarrow (N', o'))$ if, and only if, α factorizes over $(M_x, \overline{o})$.

We define

$$M := \bigcap_{x \in S_{\overline{o}}} M_x \quad \text{(the intersection as complex spaces)}$$

$$\mathfrak{S} := \overline{\mathfrak{S}}_{|M} \quad \text{(the restriction of } \mathfrak{S} \text{ to } M),$$

$$\pi := \overline{\pi}_{|\mathfrak{S}} : \mathfrak{S} \longrightarrow M \quad \text{(the restriction of } \overline{\pi} := \overline{\mathfrak{S}} \longrightarrow \overline{M} \text{ to } \mathfrak{S}),$$

$$o := \overline{o}, \ \ \phi := \overline{\phi}$$

Then $(\mathfrak{S}, \pi, M, o, \phi)$ is a complex analytic family of locally trivial deformations of S. We claim the family $(\mathfrak{S}, \pi, M, o, \phi)$ enjoys the properties (ii) and (iii) in the theorem. Indeed, by the definition of $(\mathfrak{S}, \pi, M, o, \phi)$, it is complete at $o \in M$ with respect to locally trivial deformations of S. Besides, locally trivial deformation has a good deformatiion theory in the sense of J. Bingener and H. Flenner. Therefore, by the openess property of the completeness, or versality ([5]), we conclude that $(\mathfrak{S}, \pi, M, o, \phi)$ is complete at every point $t \in M$, shrinking M sufficiently small around o if necessary. The fact that $(\mathfrak{S}, \pi, M, o, \phi)$ enjoys the property (ii) is shown as follows:

Since S is locally a complete intersection, $\mathcal{E}xt^q_{\mathcal{O}_S}(\Omega^1_S, \mathcal{O}_S) = 0$ for $q \neq 0, 1$. From this it follows that the spectral sequence

$$E_2^{p,q} = H^p(S, \mathcal{E}xt^q_{\mathcal{O}_S}(\Omega^1_S, \mathcal{O}_S)) \Longrightarrow E_\infty^{p+q} = \mathrm{Ext}^{p+q}_{\mathcal{O}_S}(\Omega^1_S, \mathcal{O}_S)$$

(cf. [8, Chapitre II, Théoremè 7.3.3]) is such that $E_2^{p,q} = 0$ for $q \neq 0, 1$. Hence we have an exact sequence

$$\cdots \longrightarrow E_2^{p-2,1} \xrightarrow{d_2} E_2^{p,0} \longrightarrow E_\infty^p \longrightarrow E_2^{p-1,1} \xrightarrow{d_2} E_2^{p+1,0} \longrightarrow \cdots$$

(cf. [8, Chapitre I, Théorème 4.6.2]). Therefore it follows that there exists an injection

$$(3.12) \qquad 0 \longrightarrow H^1(S, \Theta_S) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_S}(\Omega^1_S, \mathcal{O}_S).$$

Since the characteristic map of the family $(\mathfrak{S}, \pi, M, o, \phi)$ at $o \in M$ as a complex analytic family of locally trivial deformations of S is defined as the map $\sigma_o : T_oM \longrightarrow H^1(S, \Theta_S)$, the characteristic map $\tau_o : T_oM \longrightarrow Ext^1_{\mathcal{O}_S}(\Omega_S, \mathcal{O}_S)$ at $o \in M$ of the family $(\mathfrak{S}, \pi, M, o, \phi)$ as a complex analytic family of deformatins, not necessarily locally trivial, of S factorizes through the map $\sigma_o : \Theta_oM \longrightarrow H^1(S, \Theta_S)$. Hence, by the injectivities of $\tau_o : T_oM \longrightarrow Ext^1_{\mathcal{O}_S}(\Omega_S, \mathcal{O}_S)$ and the map in (3.12), we infer that the characteristic map $\sigma_o : T_oM \longrightarrow H^1(S, \Theta_S)$ at $o \in M$ of the family $(\mathfrak{S}, \pi, M, o, \phi)$ as a complex analytic family of locally trivial deformations of S is also injective.

Q.E.D.

Now we are going to define the modular variety $\mathcal{M}_{mix}(H^\ell(S)_\mathbb{Z})$ of mixed Hodge structures on $H^\ell(S)_\mathbb{Z} := H^\ell(S, \mathbb{Z})$ modulo torsion for $\ell = 1, 2$. We denote by

$$W[\ell] := \{\ W[\ell]_0 \subset W[\ell]_1 \subset \cdots \subset W[\ell]_\ell = H^\ell(S, \mathbb{Q})\ \} \qquad (\ell = 1, 2)$$

the weight filtration on $H^\ell(S, \mathbb{Q})$, and by

$$F := \{\ H^\ell(S, \mathbb{C}) = F^0 \supset F^1 \supset F^2 \supset \cdots \supset F^\ell\ \} \qquad (\ell = 1, 2)$$

the Hodge filtartion on $H^\ell(S, \mathbb{C})$.

We put

$$f^{(\ell),p} := \dim_\mathbb{C} F^p H^\ell(S, \mathbb{C}) \qquad (1 \le p \le \ell \quad \ell = 1, 2), \text{ and}$$

$$f^{(\ell),p}_k := \dim_\mathbb{C} F^p Gr^{W[\ell]}_k H^\ell(S, \mathbb{C}) \qquad (1 \le k \le \ell,\ 1 \le p \le k \quad \ell = 1, 2),$$

where we define

$$Gr^{W[\ell]}_k H^\ell(S, \mathbb{C}) := W[\ell]_k / W[\ell]_{k-1} \qquad (W[\ell]_{-1} = 0)$$

and

$F^p Gr^{W[\ell]}_k H^\ell(S, \mathbb{C})$:= the subspace of $Gr^{W[\ell]}_k H^\ell(S, \mathbb{C})$ corresponding to the subspace F^p of $H^\ell(S, \mathbb{C})$.

In what follows, letting V be a complex vector space and, $n_1, \cdots n_p$ natural numbers with $n_1 > n_2 > \cdots > n_p$, we use the notation

$$\begin{aligned} &\mathcal{F}lag(V; n_1, \cdots, n_p) \\ &:= \{F = (F^1, F^2, \cdots, F^p) \mid \text{sequences of decreasing complex subspaces} \\ &\quad \text{of } V, \text{ i.e.}, V \supset F^1 \supset F^2 \supset \cdots \supset F^p \text{ with } \dim_\mathbb{C} F^k = n_k \end{aligned}$$

for any k $(1 \leq k \leq p)\}$,

$Grass(V, n_1) := \mathcal{F}lag(V, n_1)$,

With this notation, we define

$\mathcal{F}_{mix}(H^2(S, \mathbb{C}))$

$:= \{ \ (F^1, F^2) \in \mathcal{F}lag\,(H^2(S, \mathbb{C}) \ ; \ f^{(2),1}, f^{(2),2}) \mid dim_{\mathbb{C}} \ F^p Gr_k^{W[2]} H^2(S, \mathbb{C})$

$= f_k^{(2),p}$ for every p, k with $1 \leq k \leq 2, \ 1 \leq p \leq k\}$,

$$\mathcal{F}(Gr_2^{W[2]} H^2(S, \mathbb{C})) := \mathcal{F}lag(Gr_2^{W[2]} H^2(S, \mathbb{C}) \ \ ; \ \ f_2^{(2),1}, f_2^{(2),2})$$

$$\mathcal{F}(Gr_1^{W[2]} H^2(S, \mathbb{C})) := Grass(Gr_1^{W[2]} H^2(S, \mathbb{C}) \ \ ; \ \ f_1^{(2),1}),$$

$$\mathcal{M}(Gr_2^{W[2]} H^2(S)_{\mathbb{Z}}) := \{(F^1, F^2) \in \mathcal{F}(Gr_2^{W[2]} H^2(S, \mathbb{C})) \mid \overline{F^1} \oplus F^2 = Gr_2^{W[2]} H^2(S, \mathbb{C})\}$$

(i.e., the modular variety of pure Hodge structures on $Gr_2^{W[2]} H^2(S)_{\mathbb{Z}}$

$:= Ker\{ \ H^2(X, \mathbb{Z}) \oplus H^2(D_S^*, \mathbb{Z}) \longrightarrow H^2(D_X^*, \mathbb{Z}) \ \}$ modulo torsion

of weight 2 [cf. Proposition 1.6])

$$\mathcal{M}(Gr_1^{W[2]} H_{\mathbb{Z}}) := \{F \in \mathcal{F}(Gr_1^{W[2]} H^2(S, \mathbb{C})) \ \mid \ \overline{F} \oplus F = Gr_1^{W[2]} H^2(S, \mathbb{C})\}$$

(the modular variety of pure Hodge structures on $Gr_1^{W[2]} H^2(S)_{\mathbb{Z}}$

$:= H^1(D_X^*, \mathbb{Z})/Im\{H^1(X, \mathbb{Z}) \oplus H^1(D_S^*, \mathbb{Z}) \longrightarrow H^1(D_X^*, \mathbb{Z})\}$ (modulo torsion)

of weight 1 (cf. Proposition 1.8)

We denote by

$$\pi_k : \mathcal{F}_{mix}(H^2(S, \mathbb{C})) \longrightarrow \mathcal{F}(Gr_k^{W[2]} H^2(S, \mathbb{C})) \ \ (k = 1, 2)$$

the map which assignes F^p to $F^p Gr_k^{W[2]} H^2(S, \mathbb{C})$ for $p = 1, 2$.

3.13 Definition. We define

$$\mathcal{M}_{mix}(H^2(S)_{\mathbb{Z}}) := \pi_1^{-1}(\mathcal{M}(Gr_1^{W[2]} H^2(S)_{\mathbb{Z}})) \cap \pi_2^{-1}(\mathcal{M}(Gr_2^{W[2]} H^2(S)_{\mathbb{Z}}))$$

and call it the *modular variety of mixed Hodge structures on* $H^2(S)_{\mathbb{Z}}$

Similarly, we define

$$\mathcal{F}_{mix}(H^1(S, \mathbb{C})) := \{ \ F \in Grass\,(H^1(S, \mathbb{C}) \ ; \ f^{(1),1}) \mid dim_{\mathbb{C}} \ F Gr_1^{W[1]} H^1(S, \mathbb{C}) = f_1^{(1),1} \ \}$$

$$\mathcal{F}(Gr_1^{W[1]} H^1(S,\mathbb{C})) := Grass\,(Gr_1^{W[1]} H^1(S,\mathbb{C})\ ;\ f_1^{(1),1})$$

$$\begin{aligned}\mathcal{M}_{mix}(Gr_1^{W[1]} H^1(S,\mathbb{Z})) := \{\, F \in \mathcal{F}(Gr_1^{W[1]} H^1(S,\mathbb{C})) \\ \mid \overline{F} \oplus F = Gr_1^{W[1]} H^1(S,\mathbb{C})\}\end{aligned}$$

(i.e., the modular variety of pure Hodge structures on $Gr_1^{W[1]} H^1(S)_{\mathbb{Z}}$ $:= Ker\{\, H^1(X,\mathbb{Z}) \oplus H^1(D_S^*,\mathbb{Z}) \longrightarrow H^1(D_X^*,\mathbb{Z})\,\}$ *modulo* torsion ([cf. Proposition 1.8])

We denote by

$$\pi_1 : \mathcal{F}_{mix}(H^1(S,\mathbb{C})) \longrightarrow \mathcal{F}(Gr_1^{W[1]} H^1(S,\mathbb{C}))$$

the map which assignes F to $FGr_1^{W[1]} H^1(S,\mathbb{C})$.

3.14 Definition. We define

$$\mathcal{M}_{mix}(H^1(S)_{\mathbb{Z}}) := \pi_1^{-1}(\mathcal{M}(Gr_1^{W[1]} H^1(S)_{\mathbb{Z}}))$$

and call it the *modular variety of mixed Hodge structures on* $H^1(S)_{\mathbb{Z}}$

Suppose we are given an algebraic surface S and the Kuranishi family $(\mathfrak{S}, \pi, M, o, \phi)$ of locally trivial deformations of S in Theorem 3.12 which satisfies the following conditions:

(i) the parameter space M is non-singular, and

(ii) all of its fibers $S_t := \pi^{-1}(t), t \in M$, are complex projective.

Then there arizes naturally variations of mixed Hodge structures on $(R_{\mathbb{Z}}^{\ell} \otimes \mathcal{O}_M,\ \ell = 1,2)$. due to Theorem 2.2. Therefore we have a holomorphic map, which is called, *period map*

$$\Phi := \Phi_1 \times \Phi_2 : M \longrightarrow \mathcal{M}_{mix}(H^1(S)_{\mathbb{Z}}) \times \mathcal{M}_{mix}(H^2(S)_{\mathbb{Z}})$$

at least from a sufficiently small open neighborhood of o in M to the product of the *modular varieties* of mixed Hodge structures on $H^{\ell}(S)_{\mathbb{Z}} := H^{\ell}(S,\mathbb{Z})$ (modulo torsion), $\ell = 1, 2$. Now the infinitesimal mixed Torelli problem for S is formulated as follows:

<Infinitesimal mixed Torelli problem for S >

Under the setting as above, is the Jacobian map of the period map Φ at o

$$d\Phi_o : T_o M \longrightarrow T_{\Phi_1(o)}(\mathcal{M}_{mix}(H^1(S)_{\mathbb{Z}})) \oplus T_{\Phi_2(o)}(\mathcal{M}_{mix}(H^2(S)_{\mathbb{Z}}))$$

injective ?

In order to clarify the relation between the Jacobian map $d\Phi_o$ and the characteristic map $\sigma_o : T_oM \longrightarrow H^1(S, \Theta_S)$ of the family $(\mathfrak{S}, \pi, M, o, \phi)$ at the point o, we define the *sheaf* $\Theta(b_\bullet)$ *of germs of holomorphic tangent vector fields to the cubic hyper-resolution* $b_\bullet : X_\bullet \longrightarrow S$. For each $\alpha \in \mathrm{Ob}(\square_2^+)$ we denote by Θ_{X_α} the sheaf of germs of holomorphic tangent vector fields on X_α ($X_0 := S$ for $0 := (0, \cdots, 0) \in \mathrm{Ob}(\square_2^+)$, and by $\Theta(\mathcal{O}_S, \mathcal{O}_{X_\alpha})$ the sheaf of germs of $\mathcal{O}_{X_\alpha}$-valued derivations on S, i.e., $\theta \in \Theta(\mathcal{O}_S, \mathcal{O}_{X_\alpha})$ is a $\mathbb{C}$-linear map $\mathcal{O}_S \longrightarrow b_{\alpha *}\mathcal{O}_{X_\alpha}$ with the property $\theta(ab) = \theta(a)b + a\theta(b)$ for $a, b \in \mathcal{O}_S$. For each $\alpha \in \mathrm{Ob}(\square_2^+)$ we define

$$tb_\alpha : b_{\alpha *}\Theta_{X_\alpha} \longrightarrow \Theta(\mathcal{O}_S, \mathcal{O}_{X_\alpha}) \quad \text{and} \tag{3.13}$$

$$\omega b_\alpha : \Theta_S \longrightarrow \Theta(\mathcal{O}_S, \mathcal{O}_{X_\alpha}) \tag{3.14}$$

by

$$tb_\alpha(\theta) := \theta b_\alpha^* \quad \text{for} \quad \theta \in b_{\alpha *}\Theta_{X_\alpha} \quad \text{and}$$

$$\omega b_\alpha(\varphi) := b_\alpha^* \varphi \quad \text{for} \quad \varphi \in \Theta_S,$$

where $b_\alpha^* : \mathcal{O}_S \longrightarrow b_{\alpha *}\mathcal{O}_{X_\alpha}$ is the pull-back.

3.15 Definition. We define

$$\begin{aligned} &\Theta(b_\bullet) := \\ &Ker\{\oplus_{\alpha \in \mathrm{Ob}(\square_2^+)} b_{\alpha *}\Theta_{X_\alpha} \longrightarrow \oplus_{\alpha \in \mathrm{Ob}(\square_2)} \Theta(\mathcal{O}_S, \mathcal{O}_{X_\alpha}) : \\ &\qquad (\theta_\alpha) \longrightarrow tb_\alpha(\theta_\alpha) - \omega b_\alpha(\theta_0)\}, \end{aligned} \tag{3.15}$$

and call it the *sheaf of germs of holomorphic tangent vector fields to the 2-cubic hyper-resolution* $b_\bullet : X_\bullet \longrightarrow S$.

Now we are going to define the *Kodaira-Spencer map*

$$\rho_o : T_oM \longrightarrow H^1(S, \Theta(b_\bullet))$$

for the 2-cubic hyper-equisingular family $\mathfrak{X}_\bullet \xrightarrow{a_\bullet} \mathfrak{S} \xrightarrow{\pi} M$ of deformations of the cubic hyper-resolution $b_\bullet : X_\bullet \longrightarrow S$ arising from an family $(\mathfrak{S}, \pi, M, o, \phi)$ of locally trivial deformations of S. By the *analytically local triviality* of the family $\mathfrak{X}_\bullet \xrightarrow{a_\bullet} \mathfrak{S} \xrightarrow{\pi} M$, shrinking M sufficiently small around o, we may assume that there is a special system of open coverings $\mathcal{U}_\alpha := \{U_i^{(\alpha)}\}_{i \in \Lambda_\alpha}$ of X_α ($\alpha \in \mathrm{Ob}(\square_2^+), X_0 = S$) consisting of Stein coordinate neighborhoods as described in (2.7). In addition, since b_α is a finite map for every $\alpha \in \mathrm{Ob}(\square_2)$, we may assume that the system of open coverings $\mathcal{U}_\alpha := \{U_i^{(\alpha)}\}_{i \in \Lambda_\alpha}$ ($\alpha \in \mathrm{Ob}\square_2^+$) satisfies the following condition;

(v) for each pair of $\alpha \in \mathrm{Ob}(\square_2^+)$ and $i \in \Lambda_0$, there exists a finite subset $\Lambda_\alpha(i)$ of

Λ_α such that:

$$(3.16)\qquad \begin{aligned}&(a)\ b_\alpha^{-1}(U_i^{(0)}) = \cup_{j\in\Lambda_\alpha(i)} U_j^{(\alpha)}, \text{ and}\\ &(b)\ U_j^{(\alpha)} \cap U_k^{(\alpha)} = \emptyset \text{ for } \quad j,k \in \Lambda_\alpha(i) \text{ with } j \neq k.\end{aligned}$$

We take such a special system of open coverings and fix it. In what follows we will always calculate with respect to this coverings unless otherwise mentioned. For each $\alpha \in \mathrm{Ob}(\square_2^+)$ we denote by $\mathfrak{C}^p(\mathcal{U}_\alpha, \Theta_{X_\alpha})$ (resp. $\mathfrak{Z}^p(\mathcal{U}_\alpha, \Theta_{X_\alpha})$) the p-th Cĕch cochains (resp. the p-th Cĕch cocycles) with values in the sheaf Θ_{X_α} with respect to the Stein covering $\mathcal{U}_\alpha$, and by

$$\delta_\alpha^{(p)} : \mathfrak{C}^p(\mathcal{U}_\alpha, \Theta_{X_\alpha}) \to \mathfrak{C}^{p+1}(\mathcal{U}_\alpha, \Theta_{X_\alpha})$$

Cĕch coboundary map. We define a subcomplex $\mathfrak{C}^p(b_\bullet)$ of $\oplus_{\alpha\in\mathrm{Ob}(\square_2^+)} \mathfrak{C}^p(\mathcal{U}_\alpha, \Theta_{X_\alpha})$ by

$$(3.17)\qquad \begin{aligned}&\mathfrak{C}^p(b_\bullet) :=\\ &Ker\{\oplus_{\alpha\in\mathrm{Ob}(\square_2^+)} \mathfrak{C}^p(\mathcal{U}_\alpha, \Theta_{X_\alpha})\\ &\qquad \xrightarrow{\oplus_{\alpha\in\mathrm{Ob}(\square_2)}(tb_\alpha - \omega b_\alpha)} \oplus_{\alpha\in\mathrm{Ob}(\square_2)} C^p(\mathcal{U}_0, \Theta(\mathcal{O}_S, \mathcal{O}_{X_\alpha}))\}\end{aligned}$$

Then by the definition we have the following commutative diagram:

$$(3.18)\qquad \begin{array}{ccc} 0 \to \mathfrak{C}^p(\mathcal{U}_0, \Theta(b_\bullet)) & \to & \oplus_{\alpha\in\mathrm{Ob}(\square_2^+)} \mathfrak{C}^p(b_\alpha^{-1}(\mathcal{U}_0), \Theta_{X_\alpha}) \\ \downarrow & & \downarrow \\ 0 \to \mathfrak{C}^p(b_\bullet) & \longrightarrow & \oplus_{\alpha\in\mathrm{Ob}(\square_2^+)} \mathfrak{C}^p(\mathcal{U}_\alpha, \Theta_{X_\alpha}) \\ & \xrightarrow{\oplus_{\alpha\in\mathrm{Ob}(\square_2)}(tb_\alpha - \omega b_\alpha)} & \oplus_{\alpha\in\mathrm{Ob}(\square_2)} \mathfrak{C}^p(\mathcal{U}_0, \Theta(\mathcal{O}_S, \mathcal{O}_{X_\alpha})) \\ & & \| \\ & \xrightarrow{\oplus_{\alpha\in\mathrm{Ob}(\square_2)}(tb_\alpha - \omega b_\alpha)} & \oplus_{\alpha\in\mathrm{Ob}(\square_2)} \mathfrak{C}^p(\mathcal{U}_0, \Theta(\mathcal{O}_S, \mathcal{O}_{X_\alpha})), \end{array}$$

where $b_\alpha^{-1}(\mathcal{U}_0) := \{b_\alpha^{-1}(\mathcal{U}_i^{(0)})\}_{i\in\Lambda_0}$. Note that this commutative diagram commutes with Cĕch coboundary maps.

Let $(t_1, \cdots, t_m)$ and $(x_{i1}^{(\alpha)}, \cdots, x_{in_\alpha}^{(\alpha)})$ ($\alpha \in \mathrm{Ob}(\square_2^+), i \in \Lambda_\alpha, n_\alpha := dim\, X_\alpha$ for $\alpha \in \mathrm{Ob}(\square_2), n_0 = 3$, the local embedding dimension of $X_0 = S$) be local coordinate systems on M and $U_i^{(\alpha)}$, respectively. (For $X_0 = S$ we take a local embedding $S \subset \mathbb{C}^3$ at each point of S and consider the ploblem modulo $\mathcal{I}(S)$, the ideal sheaf of S in $\mathcal{O}_{\mathbb{C}^3}$.) Then $(x_{i1}^{(\alpha)}, \cdots, x_{in_\alpha}^{(\alpha)}, t_1, \cdots, t_m)$ constitutes a local coordinate system in $V_i^{(\alpha)} := U_i^{(\alpha)} \times M$. We denote by

$$\begin{cases} x_{i\mu}^{(\alpha)} = & \varphi_{ij}^{(\alpha)\mu}(x_{j1}^{(\alpha)}, \cdots, x_{jn_\alpha}^{(\alpha)}, t_1, \cdots, t_m) \ \ (1 \le \mu \le n_\alpha) \\ t_\xi = t_\xi & (1 \le \xi \le m) \end{cases}$$

the transition functions of local coordinate systems in $U_i^{(\alpha)} \cap U_j^{(\alpha)}$ for $i, j \in \Lambda_\alpha$ with $U_i^{(\alpha)} \cap U_j^{(\alpha)} \neq \emptyset$. They satisfy the compatibility conditions:

$$\begin{aligned} &\varphi_{ik}^{(\alpha)\mu}(x_{k1}^{(\alpha)}, \cdots, x_{kn_\alpha}^{(\alpha)}, t) \\ &= \varphi_{ij}^{(\alpha)\mu}(\varphi_{jk}^{(\alpha)1}(x_{k1}^{(\alpha)}, \cdots, x_{kn_\alpha}^{(\alpha)}, t), \cdots, \varphi_{jk}^{(\alpha)n_\alpha}(x_{k1}^{(\alpha)}, \cdots, x_{kn_\alpha}^{(\alpha)}, t), t). \end{aligned}$$

Hence

$$\frac{\partial \varphi_{ik}^{(\alpha)\mu}}{\partial t_\xi}(x_k^{(\alpha)}, t) =$$
$$\sum_{\zeta=1}^{n_\alpha} \frac{\partial \varphi_{ij}^{(\alpha)\mu}}{\partial x_j^{(\alpha)\zeta}}(\varphi_{jk}^{(\alpha)}(x_k^{(\alpha)}, t), t) \frac{\partial \varphi_{jk}^{(\alpha)\zeta}}{\partial t_\xi}(x_k^{(\alpha)}, t) + \frac{\partial \varphi_{ij}^{(\alpha)\mu}}{\partial t_\xi}(\varphi_{jk}^{(\alpha)}(x_k^{(\alpha)}, t), t)$$

This implies that if we define

$$\theta_{ik}^{\alpha} := \sum_{\mu=1}^{n_\alpha} \sum_{\xi=1}^{m} b_\xi \frac{\partial \varphi_{ik}^{(\alpha)\mu}}{\partial t_\xi}(x_k^{(\alpha)}, 0)(\frac{\partial}{\partial x_{i\mu}^{(\alpha)}})$$

for

$$\tau = \sum_{\xi=1}^{m} b_\xi (\frac{\partial}{\partial t_\xi})_o \in T_o M,$$

then

$$\theta_\alpha := \{\theta_{ik}^{\alpha}\}_{i,k \in \Lambda_\alpha} \in \mathfrak{Z}^1(\mathcal{U}_\alpha, \Theta_{X_\alpha}).$$

On each $U_i^{(\beta)}$ $(i \in \Lambda_\beta)$ we express the holomorphic map $E_{\alpha\beta} : \mathfrak{X}_\beta \to \mathfrak{X}_\alpha$ corresponding to an arrow $\alpha \to \beta$ in $\mathrm{Ob}(\square_2)$ as

$$\begin{cases} x_{\lambda_{\alpha\beta}(i)\mu}^{(\alpha)} = & e_{\alpha\beta,\mu}^{i}(x_{i1}^{(\beta)}, \cdots, x_{in_\beta}^{(\beta)}) \ \ (1 \le \mu \le n_\alpha) \\ t_\xi = t_\xi & (1 \le \xi \le m) \end{cases}$$

They satisfy the compatibility conditions:

$$\begin{aligned} &\varphi_{ik}^{(\alpha)\mu}(e_{\alpha\beta,1}^{k}(x_k^{(\beta)}), \cdots, e_{\alpha\beta,n_\alpha}^{k}(x_k^{(\beta)}), t) \\ &\qquad = e_{\alpha\beta,\mu}^{i}(\varphi_{ik}^{(\beta)}(x_k^{(\beta)}, t)) \quad (0 \le \mu \le n_\alpha). \end{aligned}$$

Hence

$$\frac{\partial \varphi_{ik}^{(\alpha)\mu}}{\partial t_\xi}(e^k_{\alpha\beta,1}(x_k^{(\beta)}), \cdots, e^k_{\alpha\beta,n_\alpha}(x_k^{(\beta)}), t)$$
$$= \sum_{\zeta=1}^{n_\beta} \frac{\partial e^i_{\alpha\beta,\mu}}{\partial x_i^{(\beta)\zeta}}(\varphi_{ik}^{(\beta)}(x_k^{(\beta)}, t), t)\frac{\partial \varphi_{ik}^{(\beta)\zeta}}{\partial t_\xi}(x_k^{(\beta)}, t)$$

This means

$$de_{\alpha\beta}(\theta_\beta) = e^*_{\alpha\beta}(\theta_\alpha).$$

Hence

$$\{\theta_\alpha\}_{\alpha\in(\square_2^+)} \in \mathfrak{Z}^1(b_\bullet), \tag{3.19}$$

where $\mathfrak{Z}^1(b_\bullet)$ stands for the 1-cocycle group of the complex $\mathfrak{C}^\bullet(b_\bullet)$ defined in (3.17). The θ_α in fact defines an element of $\mathfrak{C}^1(b_\alpha^{-1}(\mathcal{U}_0), \Theta_{X_\alpha})$ for each $\alpha \in \mathrm{Ob}(\square_2)$ due to the condition (v) in (3.16). Hence by (3.18)

$$\{\theta_\alpha\}_{\alpha\in \mathrm{Ob}(\square_2^+)} \in \mathfrak{Z}^1(\mathcal{U}_0, \Theta(b_\bullet))$$

We define a map $\rho_0 : T_oM \longrightarrow H^1(S, \Theta(b_\bullet))$ by

$$\rho_0(\tau) := \{\theta_\alpha\}_{\alpha\in \mathrm{Ob}(\square_2^+)} \in \check{H}^1(\mathcal{U}_0, \Theta(b_\bullet)) \text{ (Cĕch cohomology)} \tag{3.20}$$
$$\simeq H^1(S, \Theta(b_\bullet))$$

for $\tau \in T_oM$. We can see that the map $\check{\rho}$ thus defined is independent of the choice of a system of open coverings $\{\mathcal{U}_\alpha\}_{\alpha\in \mathrm{Ob}(\square_2^+)}$ of $b_\bullet : X_\bullet \longrightarrow S$, subject to the conditions in (2.7) and the condition (v) in (3.16).

3.16 Definition. We call the map ρ thus defined the *characteristic map, or Kodaira-Spencer map at o of the 2-cubic hyper-equisingular family $\mathfrak{X}_\bullet \xrightarrow{a_\bullet} \mathfrak{S} \xrightarrow{\pi} M$ of deformations of the cubic hyper-resolution $b_\bullet : X_\bullet \longrightarrow S$ arising from an analytic family $(\mathfrak{S}, \pi, M, o, \phi)$ of locally trivial deformations of S.*

For $\ell = 1, 2$, let

$$F^0 := R^\ell_{\mathcal{O}}(\pi) \supset F^1R^\ell_{\mathcal{O}}(\pi) \supset \cdots \supset F^\ell R^\ell_{\mathcal{O}}(\pi) \supset F^{\ell+1}R^\ell_{\mathcal{O}}(\pi) = 0$$

$$\text{(resp. } F^0_t := H^\ell(S_t, \mathbb{C}) \supset F^1_tH^\ell(S_t, \mathbb{C}) \supset \cdots \supset F^\ell_tH^\ell(S_t, \mathbb{C})$$
$$\supset F^{\ell+1}_tH^\ell(S_t, \mathbb{C}) = 0,\ t \in M)$$

be the Hodge filtration on $R^\ell_{\mathcal{O}}(\pi)$ (resp. $H^\ell(S_t, \mathbb{C})$). We put

$$Gr^p_FR^\ell_{\mathcal{O}}(\pi) := F^pR^\ell_{\mathcal{O}}(\pi)/F^{p+1}R^\ell_{\mathcal{O}}(\pi), \quad (0 \le p \le \ell)$$

*$(Gr_F^p H^\ell(S_t,\mathbb{C}) := F_t^p H^\ell(S_t,\mathbb{C})/F_t^{p+1}H^\ell(S_t,\mathbb{C}),\quad (t\in M,$
$0\le p\le \ell))$.*

The relation between the Jacobian map $(d\Phi)_o$ and the characteristic map $\rho_o : T_oM \to H^1(S_o,\Theta(b_\bullet))$ is given by the following theorem.

3.17 Theorem.

(i) *For a vector $\tau \in T_oM$, $d\Phi_o(\tau)$ belongs to the subspace*

$$\oplus_{\ell=1}^2\{\oplus_{p=1}^\ell \mathrm{Hom}_{\mathbb{C}}(Gr_{F_o}^p H^\ell(S_o,\mathbb{C}), Gr_{F_o}^{p-1}H^\ell(S_o,\mathbb{C}))\}$$

of $T_{\Phi_1(o)}(\mathcal{M}_{mix}(H^1(S)_{\mathbb{Z}}))\oplus T_{\Phi_2(o)}(\mathcal{M}_{mix}(H^2(S)_{\mathbb{Z}}))$
$\simeq \oplus_{\ell=1}^2\{\oplus_{p=1}^\ell \mathrm{Hom}_{\mathbb{C}}(Gr_{F_o}^p(H^\ell(S_o,\mathbb{C}), H^\ell(S_o,\mathbb{C})/F_o^p(H^\ell(S_o,\mathbb{C}))\}$.

(ii) *For every pair of integers (ℓ,p) with $1\le \ell\le 2$, $1\le p\le \ell$, there is an isomorphism*

$$\mathrm{Hom}_{\mathbb{C}}(Gr_{F_o}^p(H^\ell(S_o,\mathbb{C}), Gr_{F_o}^{p-1}(H^\ell(S_o,\mathbb{C}))$$
$$\simeq \mathrm{Hom}_{\mathbb{C}}(\mathbb{H}^{\ell-p}(\Omega_{X_\bullet}^p[1]), \mathbb{H}^{\ell-p+1}(\Omega_{X_\bullet}^{p-1}[1]),$$

where we simply denote $H^i(s(\oplus_{\alpha\in \mathrm{Ob}(\square_2)}C^\bullet(\mathcal{U}_\alpha,\Omega_{X_\alpha}^k))[1])$ by $\mathbb{H}^i(\Omega_{X_\bullet}^k[1])$.

(iii) *Any element of $H^1(S_o,\Theta(b_\bullet))$ defines an element of*

$$\mathrm{Hom}_{\mathbb{C}}(\mathbb{H}^{\ell-p}(\Omega_{X_\bullet}^p[1])), \mathbb{H}^{\ell-p+1}(\Omega_{X_\bullet}^{p-1}[1])))$$

by the coupling through the contraction.

(iv) *The following diagram commutes up to $\oplus_{\ell=1}^2\{\oplus_{p=1}^\ell(-1)^{p+1}\}$:*

$$\begin{array}{ccc} T_oM & \xrightarrow{(d\Phi)_o} & \oplus_{\ell=1}^2\{\oplus_{p=1}^\ell Hom_{\mathbb{C}}(Gr_{F_o}^p(H^\ell(S_o,\mathbb{C}), Gr_{F_o}^{p-1}H^\ell(S_o,\mathbb{C}))\} \\ & \searrow^{\rho_o} \qquad \nearrow_{\tau_o} & \\ & H^1(S_o,\Theta(b_\bullet)) & \end{array}$$

where the map τ_o is the one induced by the coupling through the contraction and the isomorphism in (ii).

For the proof of this theorem we prepare a lemma.

3.18 Lemma. *Let $f : X \longrightarrow Y$ be a holomorphic map between complex manifolds and let (θ_X, θ_Y) be a holomorphic vector field of a holomorphic map $f : X \longrightarrow Y$; that is, θ_X and θ_Y are holomorphic vector fields on X and Y, respectively, which are subject to the requirement that $df(\theta_X) = f^*(\theta_Y)$ in $\Gamma(X, f^*\Theta_Y)$, where $f^*\Theta_Y$ denotes the inverse image of the sheaf of germs of holomorphic vector fields on Y. Then we have*

$$\theta_X \lfloor f^*\omega = f^*(\theta_Y \lfloor \omega) \tag{3.21}$$

for any holomorphic form ω on Y, where $\lfloor$ stands for the contraction.

Proof. Let m and n be the dimensions of X and Y, respectively. In terms of local coordinates we express f, θ_X and θ_Y as follows:

$$\begin{aligned} y_i &= f_i(x_1, \cdots, x_n) \quad (1 \le i \le m), \\ \theta_X &= \sum_{i=1}^{n} a_i(x)(\frac{\partial}{\partial x_i}) \text{ and} \\ \theta_Y &= \sum_{j=1}^{m} b_j(y)(\frac{\partial}{\partial y_j}). \end{aligned}$$

Then the condition $df(\theta_X) = f^*(\theta_Y)$ is restated as

$$\sum_{i=1}^{n} a_i(x)\frac{\partial y_j}{\partial x_i} = b_j(f(x)) \quad (1 \le j \le m). \tag{3.22}$$

It suffices to prove the assertion for a holomorphic diferential p-form of the form

$$\omega = dy_{i_1} \wedge \cdots \wedge dy_{i_p} \quad (1 \le i_1 < \cdots i_p \le m)$$

1) *The case* $p \le \dim X$:

For such a form ω we have

$$\begin{aligned} f^*\omega &= \sum_{j_1 < \cdots < j_p} \frac{\partial(y_{i_1} \cdots y_{i_p})}{\partial(x_{j_1} \cdots x_{j_p})} dx_{j_1} \wedge \cdots \wedge dx_{j_p} \quad \text{and} \\ \theta_X \lfloor f^*\omega &= \sum_{j_1 < \cdots < j_p} \frac{\partial(y_{i_1} \cdots y_{i_p})}{\partial(x_{j_1} \cdots x_{j_p})} [\theta_X \lfloor (dx_{j_1} \wedge \cdots \wedge dx_{j_p})] \\ &= \sum_{j_1 < \cdots < j_p} \frac{\partial(y_{i_1} \cdots y_{i_p})}{\partial(x_{j_1} \cdots x_{j_p})} (\sum_{\mu=1}^{p} (-1)^{\mu-1} a_{j\mu}(x) dx_{j_1} \wedge \cdots \wedge d\check{x}_{j_\mu} \wedge \cdots \wedge dx_{j_p}) \end{aligned}$$

On the other hand,

$$\theta_Y \lfloor \omega = (\sum_{\lambda=1}^{p}(-1)^{\lambda-1} b_{i\lambda}(y) dy_{j_1} \wedge \cdots \wedge d\check{y}_{i_\lambda} \wedge \cdots \wedge dy_{j_p} \quad \text{, and}$$

$$f^*(\theta_Y \lfloor \omega) =$$

$$\sum_{\lambda=1}^{p}(-1)^{\lambda-1} b_{i\lambda}(f(x))\{\sum_{j_1<\cdots<j_p} \frac{\partial(y_{i_1}\cdots \check{y}_{i_\lambda}\cdots y_{i_p})}{\partial(x_{j_1}\cdots x_{j_{p-1}})} dx_{j_1} \wedge \cdots \wedge dx_{j_{p-1}}\}$$

$$= \sum_{j_1<\cdots<j_{p-1}} (\sum_{\lambda=1}^{p}(-1)^{\lambda-1} b_{i\lambda}(f(x)) \frac{\partial(y_{i_1}\cdots \check{y}_{i_\lambda}\cdots y_{i_p})}{\partial(x_{j_1}\cdots x_{j_{p-1}})}) dx_{j_1} \wedge \cdots \wedge dx_{j_{p-1}}\}$$

$$= \sum_{j_1<\cdots<j_{p-1}} \{\sum_{\lambda=1}^{p}(-1)^{\lambda-1}$$

$$\times (\sum_{i=1}^{n} a_i(x) \frac{\partial y_{i_\lambda}}{\partial x_i}(x)) \frac{\partial(y_{i_1}\cdots \check{y}_{i_\lambda}\cdots y_{i_p})}{\partial(x_{j_1}\cdots x_{j_{p-1}})}\} dx_{j_1} \wedge \cdots \wedge dx_{j_{p-1}} \qquad \text{(by (3.9))}$$

$$= \sum_{j_1<\cdots<j_{p-1}} \{\sum_{i=1}^{n} a_i(x)$$

$$\times (\sum_{\lambda=1}^{p}(-1)^{\lambda-1} \frac{\partial y_{i_\lambda}}{\partial x_i}(x)) \frac{\partial(y_{i_1}\cdots \check{y}_{i_\lambda}\cdots y_{i_p})}{\partial(x_{j_1}\cdots x_{j_{p-1}})}\} dx_{j_1} \wedge \cdots \wedge dx_{j_{p-1}}$$

$$(3.23) \quad = \sum_{j_1<\cdots<j_{p-1}} \{\sum_{i=1}^{n} a_i(x) \frac{\partial(y_{i_1}\cdots y_{i_p})}{\partial(x_i, x_{j_1}\cdots x_{j_{p-1}})}\} dx_{j_1} \wedge \cdots \wedge dx_{j_{p-1}}$$

$$= \sum_{j_1<\cdots<j_{p-1}} \{\sum_{\mu=1}^{p}(-1)^{\mu-1} a_{j_\mu}(x) \frac{\partial(y_{i_1}\cdots y_{i_p})}{\partial(x_{j_1}\cdots x_{j_p})}\} dx_{j_1} \wedge d\check{x}_{j_\mu} \wedge \cdots \wedge dx_{j_p}).$$

Therefore we have $\theta_X \lfloor f^*\omega = f^*(\theta_Y \lfloor \omega)$ as asserted.

2) *The case* $p > \dim X$:

In this case we have $f^*\omega = 0$, and so the left-hand-side in (3.21) vanishes. The right-hand-side in (3.21) also vanishes, because the index i in the expression in (3.23) appears at least once among the indexes $j_1, \cdots, j_{p-1}$. Therefore the equality in (3.21) holds.

Q.E.D.

Proof of Theorem 3.17.

Since the problem is local with respect to M, shrinking M sufficiently small around the point o, we may assume that $a_\bullet : \mathfrak{X}_\bullet \to \mathfrak{S}$ is covered by a system of open coverings $\{\mathcal{V}_\alpha\}_{\alpha\in \mathrm{Ob}(\Box_2^+)}$ consisting of Stein coordinates neighborhoods subject to the conditions (i) through (iv) in (2.7) and (v) in (3.16). In what follows we shall use the same notation as in §2.

(i) By Theorem 2.8, Theorem 2.9 and (2.5), we have

$$\begin{aligned} R^\ell_{\mathcal{O}}(\pi) := R^\ell\pi_*(\pi^{\cdot}\mathcal{O}_M) &\simeq \mathbb{R}^\ell(DR^\bullet_{\mathfrak{S}/M}) \\ &\simeq \mathbb{R}^\ell\pi_*(s(a_{1\bullet *}\Omega^\bullet_{\mathfrak{X}_\bullet/M})[1]) \\ &\simeq H^\ell(s(\oplus_{\alpha\in \mathrm{Ob}(\Box_2)}\mathrm{tot}\,\mathfrak{C}^\bullet(\mathcal{V}_\alpha, \Omega^\bullet_{\mathfrak{X}_\alpha/M}))[1]) \end{aligned}$$

By the Griffiths transversality (Theorem 2.10 (iii)), the Gauss-Manin connection ∇ on $R^\ell_{\mathcal{O}}(\pi)$ induce the map:

$$Gr^p_F(\nabla) : Gr^p_F R^\ell_{\mathcal{O}_M}(\pi) \longrightarrow \Omega^1_M \otimes Gr^{p-1}_F R^\ell_{\mathcal{O}_M}(\pi)$$

which is called the *quotient of Gauss-Manin connection*. For a vector $\tau \in T_oM$, we take a local holomorphic vector field $\tilde{\tau}$ on M around o with $\tilde{\tau}_{|t=0} = \tau$. We define

$$\tilde{\tau}\cdot Gr^p_F(\nabla) \in \mathrm{Hom}_{\mathcal{O}_M}(Gr^p_F R^\ell_{\mathcal{O}}(\pi), Gr^{p-1}_F R^\ell_{\mathcal{O}}(\pi)) \tag{3.24}$$

by the contraction of $Gr^p_F(\nabla)(s)$ by $\tilde{\tau}$ for a local cross-section s of $Gr^p_F R^\ell_{\mathcal{O}}(\pi)$. Then $d\Phi_o(\tau)$ is nothing but $\oplus^\ell_{p=1}\tilde{\tau}\cdot Gr^p_F(\nabla)_{|t=0}$, and the assertion (i) of the theorem follows from (3.24)

(ii) The assertion (ii) of the theorem follows from Theorem 2.10 (i).

(iii) Let $\theta = \{\theta_\alpha\}_{\alpha\in \mathrm{Ob}(\Box_2^+)} \in \mathfrak{Z}^1(b_\bullet)$ be a 1-cocycle of the complex $\mathfrak{C}^\bullet(b_\bullet)$ where $\theta_\alpha \in \mathfrak{Z}^1(\mathcal{U}_\alpha, \Theta_{X_\alpha})$ for $\alpha \in \mathrm{Ob}(\Box_2^+)$, and let $\omega = \{\omega_\alpha\} \in \mathfrak{Z}^{\ell-p+1}((\Omega^p_{X_\cdot})$ be a $(\ell-p+1)$- cocycle of the complex $K^\bullet(\Omega^p_{X_\bullet}) = s(\oplus_{\alpha\in \mathrm{Ob}(\Box_2)}\mathfrak{C}^\bullet(\mathcal{U}_\alpha, \Omega^p_{X_\alpha}))$ where $\omega_\alpha \in \mathfrak{C}^{\ell-p+1-|\alpha|}(\mathcal{U}_\alpha, \Omega^p_{X_\alpha})$ for $\alpha \in \mathrm{Ob}(\Box_2)$ with $1 \le |\alpha| \le \ell-p+1$. We define

$$\theta\lfloor\omega := \{\theta_\alpha\lfloor\omega_\alpha\} \in K^{\ell-p+2}(\Omega^{p-1}_{X_\bullet}) := s(\oplus_{\alpha\in \mathrm{Ob}(\Box_2)}\mathfrak{C}^\bullet(\mathcal{U}_\alpha, \Omega^{p-1}_{X_\alpha}))^{\ell-p+2})$$

where $\theta_\alpha\lfloor\omega_\alpha \in \mathfrak{C}^{\ell-p+2-|\alpha|}(\mathcal{U}_\alpha, \Omega^{p-1}_{X_\alpha})$ is defined by

$$(\theta_\alpha\lfloor\omega_\alpha)(i_0, i_1, \cdots, i_{\ell-p+2-|\alpha|}) := \theta_\alpha(i_0, i_1)\lfloor\omega_\alpha(i_1, \cdots, i_{\ell-p+2-|\alpha|}).$$

What we have to verify is that

$$\theta\lfloor\omega \in \mathfrak{Z}^{\ell-p+2}(\Omega^{p-1}_{X_\bullet}).$$

In order to verify this, we will show that

$$D^{(\ell-p+2)}(\theta\lfloor\omega)(\alpha; p-1; i_0\cdots i_r) \equiv 0$$

for any multi-index $(\alpha; p-1; i_0 \cdots i_r)$ with $|\alpha|+r = \ell-p+2$, $\alpha = (\alpha_0, \alpha_1, \alpha_2) \in \mathrm{Ob}(\Box_2)$ where

$$(3.25)\quad \begin{aligned} D^{(\ell-p+2)} : K^{\ell-p+2}(\Omega^{p-1}_{X_\bullet}) &:= \oplus_{|\alpha|+r=\ell-p+2}\mathfrak{C}^r(\mathcal{V}_\alpha, \Omega^{p-1}_{X_\alpha}) \\ \to K^{\ell-p+3}(\Omega^{p-1}_{X_\bullet}) &:= \oplus_{|\alpha|+r=\ell-p+3}\mathfrak{C}^r(\mathcal{V}_\alpha, \Omega^{p-1}_{X_\alpha}) \end{aligned}$$

is the total differential of the complex $K^\bullet(\Omega^{p-1}_{X_\bullet})$ defined by

$$D^{(\ell-p+2)} := \oplus_{|\alpha|+r=\ell-p+2}\{\sum_{\substack{1\le j\le 3\\ \alpha+e_j\in \mathrm{Ob}(\Box^2)}} (-1)^{\varepsilon_j} d^{(p-1,r)*}_{\alpha,j} + (-1)^{|\alpha|}\delta^{(p,q)}_\alpha\}$$

(cf. (2.14)).

First, we have

$$\begin{aligned}
&D^{(\ell-p+2)}(\theta\lfloor\omega)(\alpha; p-1; i_0\cdots i_r)\\
&= \sum_{\substack{1\le j\le 3\\ \alpha-e_j\in \mathrm{Ob}(\Box^2)}} (-1)^{\varepsilon_j} e^*_{\alpha-e_j,\alpha}(\theta\lfloor\omega)(\alpha-e_j; p-1; \lambda_{\alpha-e_j,\alpha}(i_0)\cdots\lambda_{\alpha-e_j,\alpha}(i_r))\\
&\qquad +\delta(\theta\lfloor\omega)(\alpha; p-1; i_0\cdots i_r) \quad (\varepsilon_j = \alpha_0+\cdots+\alpha_{j-1})\\
&= \sum_{\substack{1\le j\le 3\\ \alpha-e_j\in \mathrm{Ob}(\Box^2)}} (-1)^{\varepsilon_j} e^*_{\alpha-e_j,\alpha}(\rho(\tau)\lfloor\omega)(\alpha-e_j; p-1; \lambda_{\alpha-e_j,\alpha}(i_0)\cdots\lambda_{\alpha-e_j,\alpha}(i_r))\\
&\qquad + (-1)^{p-1+|\alpha|}\sum_{j=0}^r(-1)^j(\theta\lfloor\omega)(\alpha; p-1; i_0\cdots \check{i}_j\cdots i_r)\\
&= \sum_{\substack{1\le j\le 3\\ \alpha-e_j\in \mathrm{Ob}(\Box^2)}} (-1)^{\varepsilon_j} e^*_{\alpha-e_j,\alpha}(\theta_{\alpha-e_j}(\lambda_{\alpha-e_j,\alpha}(i_0), \lambda_{\alpha-e_j,\alpha}(i_1))\\
&\qquad\qquad \lfloor\omega(\alpha-e_j; p\ ; \lambda_{\alpha-e_j,\alpha}(i_1)\cdots\lambda_{\alpha-e_j,\alpha}(i_r)))\\
&\qquad + (-1)^{p-1+|\alpha|}\{\theta_\alpha(i_1,i_2)\lfloor\omega(\alpha; p\ ; i_2\cdots i_r) - \theta_\alpha(i_0,i_2)\lfloor\omega(\alpha;p;i_2\cdots i_r)\\
&\qquad + \sum_{j=2}^r(-1)^j\theta_\alpha(i_0,i_1)\lfloor\omega(\alpha; p\ .; i_1\cdots\check{i}_j\cdots i_r)\}\\
&= \sum_{\substack{1\le j\le 3\\ \alpha-e_j\in \mathrm{Ob}(\Box^2)}} (-1)^{\varepsilon_j}\theta_\alpha(i_0,i_1)\lfloor e^*_{\alpha-e_j,\alpha}\omega(\alpha-e_j; p\ ; \lambda_{\alpha-e_j,\alpha}(i_1)\cdots\lambda_{\alpha-e_j,\alpha}(i_r))
\end{aligned}$$

$$(3.26)\quad \begin{aligned}
&+(-1)^{p-1+|\alpha|}\{\theta_\alpha(i_1,i_2)\lfloor\omega(\alpha;p;i_2\cdots i_r) - \theta_\alpha(i_0,i_2)\lfloor\omega(\alpha;p;i_2\cdots i_r)\\
&\quad + \sum_{j=2}^r(-1)^j\theta_\alpha(i_0,i_1)\lfloor\omega(\alpha;p;i_1\cdots\check{i}_j\cdots i_r)\} \quad \text{(by Lemma 3.5)}
\end{aligned}$$

Next, since $\omega \in \mathfrak{Z}^{q+1}(\Omega^\bullet_{X_\bullet})$,

$$\begin{aligned}
&(D^{(\ell-p+1)}\omega)(\alpha;p;i_1\cdots i_r)\\
&= \sum_{\substack{1\le j\le 3\\ \alpha-e_j\in \mathrm{Ob}(\square^2)}} (-1)^{\varepsilon_j} e^*_{\alpha-e_j,\alpha}\omega(\alpha-e_j;p\;;\lambda_{\alpha-e_j,\alpha}(i_1)\cdots\lambda_{\alpha-e_j,\alpha}(i_r))\\
&\qquad\qquad + (-1)^{|\alpha|}(\delta\omega)(\alpha;p;i_1\cdots i_r)\\
&= \sum_{\substack{1\le j\le 3\\ \alpha+e_j\in \mathrm{Ob}(\square^2)}} (-1)^{\varepsilon_j} e^*_{\alpha-e_j,\alpha}\omega(\alpha-e_j;p\;;\lambda_{\alpha-e_j,\alpha}(i_1)\cdots\lambda_{\alpha-e_j,\alpha}(i_r))\\
&\qquad\qquad + (-1)^{p+|\alpha|}\sum_{j=1}^{r}(-1)^{j-1}\omega(\alpha;p\;;i_1\cdots\check{i}_j\cdots i_r)\equiv 0.
\end{aligned}$$

Hence

$$\begin{aligned}
&\textstyle\sum_{\substack{1\le j\le 3\\ \alpha-e_j\in \mathrm{Ob}(\square^2)}} (-1)^{\varepsilon_j} e^*_{\alpha-e_j,\alpha}\omega(\alpha-e_j;p\;;\lambda_{\alpha-e_j,\alpha}(i_1)\cdots\lambda_{\alpha-e_j,\alpha}(i_r))\\
&= (-1)^{p+|\alpha|}\textstyle\sum_{j=1}^{r}(-1)^{j}\omega(\alpha;p\;;i_1\cdots\check{i}_j\cdots i_r).
\end{aligned}\tag{3.27}$$

Substituting (3.27) into (3.26), we have

$$\begin{aligned}
&(-1)^{p+|\alpha|}D^{(\ell-p+2)}(\theta\lfloor\omega)(\alpha;p-1;i_0\cdots i_r)\\
&= \theta_\alpha(i_0,i_1)\lfloor(\sum_{j=1}^{r}(-1)^j\omega(\alpha;p\;;i_1\cdots\check{i}_j\cdots i_r)\}\\
&\quad - \{\theta_\alpha(i_1,i_2)\lfloor\omega(\alpha;p\;;i_2\cdots i_r) - \theta_\alpha(i_0,i_2)\lfloor\omega(\alpha;p\;;i_2\cdots i_r)\\
&\qquad + \theta_\alpha(i_0,i_1)\lfloor(\sum_{j=2}^{r}(-1)^j\omega(\alpha;p\;;i_1\cdots\check{i}_j\cdots i_r)\}\\
&= -\theta_\alpha(i_0,i_1)\lfloor\omega(\alpha;p\;;i_2\cdots i_r) - \theta_\alpha(i_1,i_2)\lfloor\omega(\alpha;p\;;i_2\cdots i_r)\\
&\qquad\qquad + \theta_\alpha(i_0,i_2)\lfloor\omega(\alpha;p\;;i_2\cdots i_r)\\
&= -(\delta\theta_\alpha)(i_0,i_1,i_2)\lfloor\omega(\alpha;p\;;i_2\cdots i_r)\equiv 0
\end{aligned}$$

as required.

(iv) Let us recall that

$$\begin{aligned}
Gr^p_F R^\ell_{\mathcal{O}}(\pi) &\simeq H^\ell(K^\bullet(F^p(\Omega^\bullet_{\mathfrak{X}_\bullet/M})[1])/K^\bullet(F^{p+1}(\Omega^\bullet_{\mathfrak{X}_\bullet/M}))[1]))\\
&\simeq H^\ell(K^\bullet(\Omega^p_{\mathfrak{X}_\bullet/M})[p+1])\\
&\simeq H^{\ell-p}(K^\bullet(\Omega^p_{\mathfrak{X}_\bullet/M})[1]),
\end{aligned}$$

where $K^\bullet(\Omega^p_{\mathfrak{X}_\bullet/M}) := s(\oplus_{\alpha\in \mathrm{Ob}(\square_2)}\mathfrak{C}^\bullet(\mathcal{V}_\alpha, \Omega^p_{\mathfrak{X}_\alpha/M}))$, and that the Gauss-Manin connection

$$\nabla : F^pR^\ell_{\mathcal{O}}(\pi) \to \Omega^1_M \otimes F^{p-1}R^\ell_{\mathcal{O}}(\pi)$$

is induced by

$$\begin{aligned}
&K^{\ell+1}(F^p(\Omega^\bullet_{\mathfrak{X}_\bullet/M})) \simeq K^{\ell+1}(G^0(F^p(\Omega^\bullet_{\mathfrak{X}_\bullet}))/G^1(F^p(\Omega^\bullet_{\mathfrak{X}_\bullet})))\\
&\xrightarrow{\phi} K^{\ell+1}(G^0(F^p(\Omega^\bullet_{\mathfrak{X}_\bullet}))) \xrightarrow{L_M+\lambda} K^{\ell+2}(G^1(F^p(\Omega^\bullet_{\mathfrak{X}_\bullet}))\\
&\to \Gamma(M,\Omega^1_M)\otimes K^{\ell+1}(G^0(F^{p-1}(\Omega^\bullet_{\mathfrak{X}_\bullet})))\\
&\simeq \Gamma(M,\Omega^1_M)\otimes K^{\ell+1}(G^0(F^{p-1}(\Omega^\bullet_{\mathfrak{X}_\bullet}))/G^1(F^{p-1}(\Omega^\bullet_{\mathfrak{X}_\bullet})))\\
&\simeq \Gamma(M,\Omega^1_M)\otimes K^{\ell+1}(F^{p-1}(\Omega^\bullet_{\mathfrak{X}_\bullet/M}))
\end{aligned}$$

Here we should notice that

$$L_M(K^{\ell+1}(G^0(F^p(\Omega^\bullet_{\mathfrak{X}_\bullet})))) \subset K^{\ell+1}(G^1(F^{p+1}(\Omega^\bullet_{\mathfrak{X}_\bullet})))$$

Hence, if we pass to the quotient

$$\begin{array}{ccc}
Gr^p_F(\nabla) : Gr^p_F R^\ell_{\mathcal{O}_M}(\pi) & \longrightarrow & \Omega^1_M \otimes Gr^{p-1}_F R^\ell_{\mathcal{O}_M}(\pi)\\
\downarrow & & \downarrow\\
H^{\ell-p}(K^\bullet(\Omega^p_{\mathfrak{X}_\bullet/M}[1])) & \longrightarrow & \Omega^1_M \otimes H^{\ell-p+1}(K^\bullet(\Omega^{p-1}_{\mathfrak{X}_\bullet/M}[1])),
\end{array}$$

L_M becomes zero map. Therefore we conclude that the quotient of Gauss-Manin connection $Gr^p_F(\nabla)$ is induced by

$$\begin{aligned}
&K^{\ell-p+1}(\Omega^p_{\mathfrak{X}_\bullet/M}) \simeq K^{\ell-p+1}(G^0(\Omega^p_{\mathfrak{X}_\bullet})/G^1(\Omega^p_{\mathfrak{X}_\bullet}))\\
&\simeq K^{\ell-p+1}(G^0(\Omega^p_{\mathfrak{X}_\bullet}))/K^{\ell-p+1}G^1(\Omega^p_{\mathfrak{X}_\bullet}))\\
&\xrightarrow{\phi} K^{\ell-p+1}(G^0(\Omega^p_{\mathfrak{X}_\bullet})) \xrightarrow{\lambda} K^{\ell-p+2}(G^1(\Omega^p_{\mathfrak{X}_\bullet}))\\
&\to K^{\ell-p+2}(G^1(\Omega^p_{\mathfrak{X}_\bullet})/K^{\ell-p+2}(G^2(\Omega^p_{\mathfrak{X}_\bullet}))\\
&\simeq K^{\ell-p+2}(G^1(\Omega^p_{\mathfrak{X}_\bullet})/G^2(\Omega^p_{\mathfrak{X}_\bullet}))\\
&\simeq \Gamma(M,\Omega^1_M)\otimes K^{\ell-p+2}(G^0(\Omega^{p-1}_{\mathfrak{X}_\bullet})/G^1(\Omega^{p-1}_{\mathfrak{X}_\bullet}))\\
&\simeq \Gamma(M,\Omega^1_M)\otimes K^{\ell-p+2}(\Omega^{p-1}_{\mathfrak{X}_\bullet})
\end{aligned} \tag{3.28}$$

We denote by

$$\hat{\lambda} : \mathfrak{Z}^{\ell-p+1}(K^\bullet(\Omega^p_{\mathfrak{X}_\bullet/M})) \to \Gamma(M,\Omega^1_M)\otimes \mathfrak{Z}^{\ell-p+2}(K^\bullet(\Omega^{p-1}_{\mathfrak{X}_\bullet/M}))$$

the map corresponding to the one in (3.28), where $\mathfrak{Z}^\bullet(K^\bullet(\Omega^p_{\mathfrak{X}_\bullet/M}))$ denote the cocycle group of the complex $K^\bullet(\Omega^p_{\mathfrak{X}_\bullet/M})$.

Claim. *For a vector* $\tau = \sum_{\gamma=1}^{m} b_\gamma (\frac{\partial}{\partial t_\gamma})_o$ *on* T_oM*, we have*

$$\rho(\tau)\lfloor\omega = (-1)^{p+1} < \tau, \ \hat{\lambda}\tilde{\omega}_{|t=0} >$$

for any $\omega \in \mathfrak{Z}^{\ell-p}(\mathcal{U}, K^\bullet(\Omega^p_{X_\bullet})[1])$*, where* $\tilde{\omega}$ *is an element of* $\mathfrak{Z}^{\ell-p}(\mathcal{V}, K^\bullet(\Omega^p_{\mathfrak{X}_\bullet/M}))$ *with* $\tilde{\omega}_{|t=o} = \omega$ *and* $< \, , \, >$ *denotes the contraction.*

Proof. By definition

$$(\hat{\lambda}\tilde{\omega})(\alpha; p\ ; i_0 \cdots i_{\ell-p+2}) = (-1)^p (I^{i_0}_\alpha - I^{i_1}_\alpha)\tilde{\omega}(\alpha; p\ ; i_1 \cdots i_{\ell-p+2})$$
$$\text{modulo } (K^{\ell-p+1}(G^2(\Omega^p_{\mathfrak{X}_\bullet})[1]),$$

where I^i_α is the *total interior product* with respect to the parameters of M (cf. (2.17)). Using local coordinates, we write
$\tilde{\omega}(\alpha; p\ ; i_1 \cdots i_{\ell-p+2})$ as

$$\tilde{\omega}(\alpha; p\ ; i_1 \cdots i_{\ell-p+2}) = \sum_{1\le\lambda_1<\cdots<\lambda_p\le n_\alpha} a_{\lambda_1\cdots\lambda_p}(x_{i_0}, t) dx^{(\alpha)\lambda_1}_{i_0} \wedge \cdots \wedge dx^{(\alpha)\lambda_p}_{i_0}$$

$(n_\alpha = dim X_\alpha)$.

Now

$$dx^{(\alpha)}_{i_0\lambda_i} = \sum_{\mu=1}^{n_\alpha} \frac{\partial \varphi^{(\alpha)\lambda_i}_{i_0 i_1}}{\partial x^{(\alpha)}_{i_1\mu}} (x^{(\alpha)}_{i_1}, t) dx^{(\alpha)}_{i_1\mu} + \sum_{\gamma=1}^{m} \frac{\partial \varphi^{(\alpha)\lambda_i}_{i_0 i_1}}{\partial t_\gamma} (x^{(\alpha)}_{i_1}, t) dt_\gamma,$$

where $x^{(\alpha)}_{i_0\lambda_i} = \partial\varphi^{(\alpha)\lambda_i}_{i_0 i_1}(x^{(\alpha)}_{i_1}, t)$ is the transition function of local coordinates on $\mathcal{V}_{i_0} \cap \mathcal{V}_{i_1}$. Hence

$$\begin{aligned} & dx^{(\alpha)}_{i_0\lambda_1} \wedge \cdots \wedge dx^{(\alpha)}_{i_0\lambda_p} \\ &= (\sum_{\gamma_1=1}^{m} \frac{\partial \varphi^{(\alpha)\lambda_1}_{i_0 i_1}}{\partial t_{\gamma_1}} dt_{\gamma_1}) \wedge dx^{(\alpha)}_{i_0\lambda_2} \wedge \cdots \wedge dx^{(\alpha)}_{i_0\lambda_p} \\ &+ dx^{(\alpha)}_{i_0\lambda_1} \wedge (\sum_{\gamma_2=1}^{m} \frac{\partial \varphi^{(\alpha)\lambda_2}_{i_0 i_1}}{\partial t_{\gamma_2}} dt_{\gamma_2}) \wedge dx^{(\alpha)}_{i_0\lambda_3} \wedge \cdots \wedge dx^{(\alpha)}_{i_0\lambda_p} \\ &+ \cdots\cdots + dx^{(\alpha)}_{i_0\lambda_1} \wedge \cdots \wedge dx^{(\alpha)}_{i_0\lambda_{p-1}} \wedge (\sum_{\gamma_p=1}^{m} \frac{\partial \varphi^{(\alpha)\lambda_p}_{i_0 i_1}}{\partial t_{\gamma_p}} dt_{\gamma_p}) \\ & \textit{modulo } (dx^{(\alpha)}_{i_1\mu_1} \wedge \cdots \wedge dx^{(\alpha)}_{i_1\lambda_p} + dt_{\gamma_1} \wedge dt_{\gamma_2}) \end{aligned}$$

Therefore, by the definition of the *interior product*, we have

$$\begin{aligned} & (\hat{\lambda}\tilde{\omega})(\alpha; p; i_0 \cdots i_{\ell-p+2}) \\ &= (-1)^{p+1} \sum_{1\le\lambda_1<\cdots<\lambda_p\le n_\alpha} a_{\lambda_1\cdots\lambda_p}(x_{i_0}, t) \\ & \quad \times \{\sum_{s=1}^{p} dx^{(\alpha)}_{i_0\lambda_1} \wedge \cdots \wedge (\sum_{\gamma_s=1}^{m} \frac{\partial \varphi^{(\alpha)\lambda_s}_{i_0 i_1}}{\partial t_{\gamma_s}} dt_{\gamma_s}) \wedge \cdots \wedge dx^{(\alpha)}_{i_0\lambda_p}\}. \end{aligned}$$

Hence, for $\tau = \sum_{\gamma=1}^{m} b_\gamma(\partial/\partial t_\gamma)$,

$$\begin{aligned} &< \tau, (\hat{\lambda}\omega)(\alpha; p; i_0 \cdots i_{q+2})_{|t=0} > \\ (3.29)\quad &= (-1)^{p+1}\{ \sum_{1\le\lambda_1<\cdots<\lambda_p\le n_\alpha} a_{\lambda_1\cdots\lambda_p}(x_{i_0}, 0) \\ &\times\{\sum_{\gamma=1}^{m} b_\gamma \\ &\times[\sum_{s=1}^{p}(-1)^{s-1}\frac{\partial\varphi_{i_0 i_1}^{(\alpha)\lambda_s}}{\partial t_\gamma}(x_{i_1}, 0)dx_{i_0\lambda_1}^{(\alpha)} \wedge\cdots\wedge d\check{x}_{i_0\lambda_s}^{(\alpha)} \wedge dx_{i_0\lambda_p}^{(\alpha)}]\}. \end{aligned}$$

On the other hand, since

$$\begin{aligned} \theta &= \{\theta_\alpha\}_{\alpha\in \mathrm{Ob}(\Box_n)}, \quad \theta_\alpha = \{\theta_{i_0,i_1}^{\alpha}\}_{i_0,i_1\in\Lambda_\alpha} \\ \theta_{i_0,i_1}^{\alpha} &= \sum_{\gamma=1}^{m}\sum_{\lambda=1}^{n_\alpha} b_\gamma(t)\frac{\partial\varphi_{i_0 i_1}^{(\alpha)\lambda}}{\partial t_\gamma}(x_{i_1}, 0)(\frac{\partial}{\partial x_{i_0\lambda}^{(\alpha)}}), \end{aligned}$$

we have

$$\begin{aligned} &(\theta\lfloor\omega)(\alpha; p-1; i_0 \cdots, i_{\ell-p+2}) = \theta_{i_0,i_1}^{\alpha}\lfloor\omega(\alpha; p\,; i_0\cdots i_{\ell-p+2}) \\ &= \{\sum_{\gamma=1}^{m}\sum_{\lambda=1}^{n_\alpha} b_\gamma \frac{\partial\varphi_{i_0 i_1}^{(\alpha)\lambda}}{\partial t_\gamma}(x_{i_1}, 0)(\frac{\partial}{\partial x_{i_0\lambda}^{(\alpha)}}))\} \\ &\quad\lfloor\{\sum_{1\le\lambda_1<\cdots<\lambda_p\le n_\alpha} a_{\lambda_1\cdots\lambda_p}(x_{i_0}, 0)dx_{i_0\lambda_1}^{(\alpha)} \wedge\cdots\wedge dx_{i_0\lambda_p}^{(\alpha)}\} \\ (3.30)\quad & \\ &= \sum_{\gamma=1}^{m} b_\gamma(t) \sum_{1\le\lambda_1<\cdots<\lambda_p\le n_\alpha} a_{\lambda_1\cdots\lambda_p}(x_{i_0}, 0) \\ &\quad\times[\sum_{s=1}^{p}(-1)^{s-1}\frac{\partial\varphi_{i_0 i_1}^{(\alpha)\lambda_s}}{\partial t_\gamma}dx_{i_0}^{(\alpha)\lambda_s}(x_{i_1}, 0)dx_{i_0\lambda_1}^{(\alpha)} \wedge\cdots\wedge d\check{x}_{i_0\lambda_s}^{(\alpha)} \wedge dx_{i_0\lambda_p}^{(\alpha)}]\}. \end{aligned}$$

Comparing (3.29) with (3.30), we conclude that $\theta\lfloor\omega = (-1)^{p+1} < \tau, \ \hat{\lambda}\omega >$.

Q.E.D. for **Claim**

This completes the proof of Theorem 3.17.

In order to explain the relation betwen the characteristic map $\rho_o : T_oM \longrightarrow H^1(S, \Theta(b_\bullet))$ of the family $\mathfrak{X}_\bullet \xrightarrow{a_\bullet} \mathfrak{S} \xrightarrow{\pi} M$ and the characteristic map $\sigma_o : T_oM \longrightarrow H^1(S, \Theta_S)$ of the family $\pi : \mathfrak{S} \longrightarrow M$, we fix some notation:

Let $S, D_S, \Sigma t_S, \Sigma \mathfrak{c}_S, X, D_X, \Sigma t_X$, D_S^*, Σt_S^*, D_X^*, Σt_X^*, $f : X \longrightarrow S$, $n_S : D_S^* \longrightarrow D_S$, $n_X : D_X^* \longrightarrow D_X$, $g : D_X^* \longrightarrow D_S^*$, $\nu_S : D_S^* \longrightarrow S$ and $\nu_X : D_X^* \longrightarrow X$ be the same as in the diagrams (1.3) and (1.4). Further, we shall use the following notation:

$\Theta_X(-\log D_X)$: the sheaf of germs of logarithmic tangent vector fields along D_X on X, i.e., the subsheaf of Θ_X consisting of derivations of $\mathcal{O}_X$ which send $\mathcal{I}(D_X)$, the ideal sheaf of D_X in $\mathcal{O}_X$, into itself.

$\Theta_{D_S^*}(-\Sigma\mathfrak{c}_S^* - \Sigma t_S^*)$: the sheaf of germs of holomorphic tangent vector on D_S^* which vanish on $\Sigma\mathfrak{c}_S^*$ and Σt_S^*, where $\Sigma\mathfrak{c}_S^*$ is the inverse images of the cuspidal point locus $\Sigma\mathfrak{c}_S$ of S by the normalization map $n_S : D_S^* \longrightarrow D_S$,

$\Theta_{D_X^*}(-\Sigma t_X^*)$: the sheaf of germs of holomorphic tangent vector fields on D_X^* which vanish on Σt_X^*. (Note that Σt_X^* coincides with the inverse image of the triple point locus Σt_S of D_S by the composed map $n_S \circ g : D_X^* \longrightarrow D_S$.)

3.19 Theorem. *There exists naturally the following exact sequence of $\mathcal{O}_S$-modules:*

$$0 \longrightarrow \Theta_S \xrightarrow{\widehat{\omega f} \oplus \widehat{\omega \nu_S}} f_*\Theta_X(-logD_X) \oplus \nu_{S*}\Theta_{D_S^*}(-\Sigma\mathfrak{c}_S^* - \Sigma t_S^*)$$

$$\xrightarrow{\widehat{\omega\nu_X} - \widehat{\omega g}} \nu_*\Theta_{D_X^*}(-\Sigma t_X^*) \longrightarrow 0 \tag{3.31}$$

where $\nu := f \circ \nu_X = \nu_S \circ g$ (cf. (1.4)).

The proof of this theorem is a direct calculation by using local coordinate description of the normalization maps $X \longrightarrow S$, $D_S^* \longrightarrow D_S$ and $D_X^* \longrightarrow D_X$, and will be accomplished after several lemmas. First we will calculate the generators of the stalk $\Theta_{S,p}$ at a cuspidal point p of S. Since the problem is local, we may assume that S is a hypersurface defined by the equation $xy^2 - z^2 = 0$ in the complex 3-space $\mathbb{C}^3$. Note that, in this case, $\Theta_S := \mathcal{H}om_{\mathcal{O}_S}(\Omega_S^1, \mathcal{O}_S)$ is isomorphic to $\Theta_{\mathbb{C}^3}/\Theta_{\mathbb{C}^3}(-\log S)$, where $\Theta_{\mathbb{C}^3}(-\log S)$ denotes the sheaf of germs of logarithmic tangent vector fields along S on $\mathbb{C}^3$, i.e., the subsheaf of $\Theta_{\mathbb{C}^3}$ consisting of derivations of $\mathcal{O}_{\mathbb{C}^3}$ which send $\mathcal{I}(S)$, the ideal sheaf of S in $\mathcal{O}_{\mathbb{C}^3}$, into itself. We define a holomorphic map $f : \mathbb{C}^2 \longrightarrow \mathbb{C}^3$ by

$$(u, v) \longrightarrow (u^2, v, uv) = (x, y, z), \tag{3.32}$$

which gives the normalization of S. Let

$$\theta = a(x, y, z)\frac{\partial}{\partial x} + b(x, y, z)\frac{\partial}{\partial y} + c(x, y, z)\frac{\partial}{\partial z}$$

be a locla holomorphic vector field defined in an open neighborhood U of the origin of $\mathbb{C}^3$, where $a(x,y,z), b(x,y,z)$ and $c(x,y,z)$ are holomorphic fuctions on U. θ is tangent to S, i.e., $\theta \cdot \mathcal{I}(S) \subset \mathcal{I}(S)$, if and only if

$$v \cdot a(u^2, v, uv) + 2u^2 \cdot b(u^2, v, uv) - 2u \cdot c(u^2, v, uv) \equiv 0$$

on $f^{-1}(U)$. We denote by $\mathcal{O}_{\mathbb{C}^2,o}$ the stalk at the origin of the structure sheaf of $\mathbb{C}^2$, by $\mathcal{O}_{\mathbb{C}^2,o}^{\oplus 3}$ the direct product of three copies of $\mathcal{O}_{\mathbb{C}^2,o}$. We define a $\mathcal{O}_{\mathbb{C}^2,o}$-submodule $\mathcal{R}_{\mathbb{C}^2,o}$ of $\mathcal{O}_{\mathbb{C}^2,o}^{\oplus 3}$ by

$$\mathcal{R}_{\mathbb{C}^2,o} := \{(\xi(u,v), \eta(u,v), \zeta(u,v)) \in \mathcal{O}_{\mathbb{C}^2,o}^{\oplus 3} | \\ v \cdot \xi(u,v) + 2u^2 \cdot \eta(u,v) - 2u \cdot \zeta(u,v) \equiv 0\}.$$

We will omit the proof of Lemma 3.20 and Lemma 3.21 below, which are direct calculations.

3.20 Lemma. *$\mathcal{R}_{\mathbb{C}^2,o}$ is generated by the elements $(u, 0, \frac{1}{2}v)$ and $(0, 1, u)$ as a $\mathcal{O}_{\mathbb{C}^2,0}$-module.*

3.21 Lemma. *For an element $\xi(u,v)$ of $\mathcal{O}_{\mathbb{C}^2,o}$ there exists an element $a(x,y,z) \in \mathcal{O}_{\mathbb{C}^3,o}$ with $\xi(u,v) = a(u^2, v, uv)$ if, and only if, $\xi(u,v)$ is of the form*

$$\xi(u,v) = \xi_0(u^2) + v \cdot \xi_1(u,v),$$

where $\xi_0(u^2)$ is a convergent power series in u^2, and $\xi_1(u,v)$ in u, v.

3.22 Lemma. *Let $\xi(u,v)$ be an element of $\mathcal{O}_{\mathbb{C}^2,o}$ which has the form*

$$\xi(u,v) = u \cdot a(u^2, v, uv), \tag{3.33}$$

where $a(x,y,z) \in \mathcal{O}_{\mathbb{C}^3,o}$. Then there exists an element $b(x,y,z) \in \mathcal{O}_{\mathbb{C}^3,o}$ with $\xi(u,v) = b(u^2, v, uv)$ if, and only if, $a(x,y,z)$ is of the form

$$a(x,y,z) = y \cdot a_1(x,y,z) + z \cdot a_2(x,y,z),$$

where $a_i(x,y,z) \in \mathcal{O}_{\mathbb{C}^3,o}$ $(i = 1, 2)$.

Proof. Assume that there exists an element $b(x,y,z) \in \mathcal{O}_{\mathbb{C}^3,o}$ with $\xi(u,v) = b(u^2, v, uv)$. Then by Lemma 3.21 $\xi(u,v)$ is expresed in the form

$$\xi(u,v) = \xi_0(u^2) + v \cdot \xi_1(u,v). \tag{3.34}$$

On the other hand we can express $a(x,y,z)$ in the form

$$a(x,y,z) = a_0(x) + y \cdot a_1(x,y,z) + z \cdot a_2(x,y,z),$$

where $a_0(x)$ is a convergent power series in x, and $a_i(x,y,z)$, $i = 1, 2,$ in x, y, z. Then by the condition (3.33) we have

$$\xi(u,v) = u \cdot a_0(u^2) + v\{u \cdot a_1(u^2, v, uv) + u^2 \cdot a_2(u^2, v, uv)\}. \tag{3.35}$$

Comparing (3.35) with (3.34), we have

$$u \cdot a_0(u^2) = \xi_0(u^2)$$

This implies $a_0(x) \equiv 0$. Thus the necessity part of the lemma has been proved. The sufficiency part is trivial.

Q.E.D.

Now we are ready to caluculate the generators of the stalk $\Theta_{S,o}$ at the cuspidal point o of S.

3.23 Lemma. *The stalk $\Theta_{S,o}$ at the cuspidal point o of S is generated by the following elements as a $\mathcal{O}_{S,o}$-module:*

$$\Theta_1 = x\frac{\partial}{\partial x} + \frac{1}{2}z\frac{\partial}{\partial z}$$
$$\Theta_2 = z\frac{\partial}{\partial x} + \frac{1}{2}y^2\frac{\partial}{\partial z}$$
$$\Theta_3 = y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$$
$$\Theta_4 = z\frac{\partial}{\partial y} + xy\frac{\partial}{\partial z}.$$

Proof. The restriction to S of the germ of a local holomorphic vector field

$$\theta = a(x,y,z)\frac{\partial}{\partial x} + b(x,y,z)\frac{\partial}{\partial y} + c(x,y,z)\frac{\partial}{\partial z}$$

at the origin of $\mathbb{C}^3$ belongs $\Theta_{S,o}$ if, and only if

$$(a(u^2,v,uv), b(u^2,v,uv), c(u^2,v,uv)) \in \mathcal{R}_{\mathbb{C}^2,o}.$$

By Lemma 3.20 this is equivalent to that there exist $\xi(u,v), \eta(u,v) \in \mathcal{O}^{\oplus 2}_{\mathbb{C}^2,o}$ such that

$$a(u^2,v,uv) = u \cdot \xi(u,v) \tag{3.36}$$

$$b(u^2,v,uv) = \eta(u,v), \quad \text{and} \tag{3.37}$$

$$c(u^2,v,uv) = \frac{1}{2}v \cdot \xi(u,v) + u \cdot \eta(u,v). \tag{3.38}$$

Substituting zero for u in (3.36), we have $a(0,v,0) \equiv 0$. Hence $a(x,y,z)$ can be written in the form

$$a(x, y, z) = x \cdot a_1(x, y, z) + z \cdot a_2(x, y, z), \tag{3.39}$$

where $a_i(x, y, z) \in \mathcal{O}_{\mathbb{C}^3, o}$, $i = 1, 2$. Then by (3.36)

$$\xi(u, v) = u \cdot a_1(u^2, v, uv) + v \cdot a_2(u^2, v, uv). \tag{3.40}$$

We put

$$a_0(x, y, z) = \frac{1}{2} z \cdot a_1(x, y, z) + \frac{1}{2} y^2 \cdot a_2(u^2, v, uv).$$

Then (3.40) implies

$$a_0(u^2, v, uv) = \frac{1}{2} v \cdot \xi(u, v).$$

By (3.38)

$$u \cdot \eta(u, v) = c(u^2, v, uv) - a_0(u^2, v, uv).$$

Hence by (3.37)

$$u \cdot b(u^2, v, uv) = c(u^2, v, uv) - a_0(u^2, v, uv).$$

Therefore, By Lemma 3.22, $b(x, y, z)$ can be written in the form

$$b(x, y, z) = y \cdot b_1(x, y, z) + z \cdot b_2(x, y, z), \tag{3.41}$$

where $b_i(x, y, z) \in \mathcal{O}_{\mathbb{C}^3, o}$, $i = 1, 2$. By (3.37) we have

$$\eta(u, v) = v \cdot b_1(u^2, v, uv) + uv \cdot b_2(u^2, v, uv) \tag{3.42}$$

Substituting (3.40) and (3.42) into (3.38), we have

$$\begin{aligned} c(u^2, v, uv) = {} & \tfrac{1}{2} uv \cdot a_1(u^2, v, uv) + \tfrac{1}{2} v^2 \cdot a_2(u^2, v, uv) \\ & + uv \cdot b_1(u^2, v, uv) + u^2 v \cdot b_2(u^2, v, uv). \end{aligned}$$

From this it follows that

$$\begin{aligned} c(x, y, z) \equiv {} & \tfrac{1}{2} z \cdot a_1(x, y, z) + \tfrac{1}{2} y^2 a_2(x, y, z) \\ & + z \cdot b_1(x, y, z) + xy \cdot b_2(x, y, z) \end{aligned} \tag{3.43}$$

holds identically on S. Cosequently, by (3.39), (3.41) and (3.43), we have

$$\begin{aligned} \theta = {} & a_1(x, y, z)(x \frac{\partial}{\partial x} + \frac{1}{2} z \frac{\partial}{\partial z}) + a_2(x, y, z)(z \frac{\partial}{\partial x} + \frac{1}{2} y^2 \frac{\partial}{\partial z}) \\ & + b_1(x, y, z)(y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}) + b_2(x, y, z)(z \frac{\partial}{\partial y} + xy \frac{\partial}{\partial z}). \end{aligned}$$

Coversely, a holomprphic vector field of this form is certainly tangent to S at the origin of $\mathbb{C}^3$.

Q.E.D.

3.24 Lemma. *With the same setting as in Theorem 3.19, let p be a cuspidal point of S.*

(i) *For any element* $\Theta \in \Theta_{S,p}$ *there exists uniquely an element* $\theta \in f_*\Theta_X(-logD_X)_p$ *such that* $tf(\theta) = \omega f(\Theta)$ *in* $\Theta(\mathcal{O}_S, \mathcal{O}_X)_p$ *(for the defintion of* $tf, \omega f$ *see (3.13) and (3.14)). Furthermore, the map which assignes to* $\Theta \in \Theta_{S,p}$ *an element* $\theta \in f_*\Theta_X(-logD_X)_p$ *with* $tf(\theta) = \omega f(\Theta)$ *is injective.*

(ii) *For any element* $\xi \in \nu_{S*}\Theta_{D_S^*}(-\Sigma \mathfrak{c}_S^* - \Sigma t_S^*)_p = \Theta_{D_S^*}(-\Sigma c_S^* - \Sigma t_S^*)_{\nu^{-1}(p)}$, *there exists uniquely* $\eta \in \nu_*\Theta_{D_X^*}(-\Sigma t_X^*)_p = \Theta_{D_X^*}(-\Sigma t_X^*)_{\nu^{-1}(p}$ *such that* $tg(\eta) = \omega g(\xi)$ *in* $\Theta(\mathcal{O}_{D_S^*}, \mathcal{O}_{D_X^*})_{\nu^{-1}(p)}$.

Proof. Since the problem is local, we may assume that S is a hypersurface defined by the equation $xy^2 - z^2 = 0$ in the complex 3-space $\mathbb{C}^3$, $X = \mathbb{C}^2$, and that f is the map defined in (3.32). We may also assume that D_S^* coincides with the double curve D_S: $y = z = 0$ of S, and that D_X^* is the inverse image of D_S by the map $f : (u, v) \longrightarrow (u^2, v, uv)$. D_X^* is defined by $v = 0$ in $\mathbb{C}^2$. That is, D_X^* is the u-axis of $\mathbb{C}^2$, D_S^* is the x-axis of $\mathbb{C}^3$, and $g : D_X^* \longrightarrow D_S^*$ is given by $u \longrightarrow u^2 = x$.

(i) Any element of $\Theta_X(-\log D_X)_{f^{-1}(p)}$ has the form

$$\theta = a(u,v)\frac{\partial}{\partial u} + v \cdot b(u,v)\frac{\partial}{\partial v}$$

where $a(u,v), b(u,v) \in \mathcal{O}_{X,f^{-1}(p)}$, we have

$$tf(\theta) = 2u \cdot a(u,v) f^*(\frac{\partial}{\partial x}) + v \cdot b(u,v) f^*(\frac{\partial}{\partial y}) \tag{3.44}$$
$$+ v\{a(u,v) + u \cdot b(u,v)\} f^*(\frac{\partial}{\partial z})$$

Here we consider $tf(\theta)$ as an element of $(f^*\Theta_{\mathbb{C}^3})_{f^{-1}(p)}$, though $tf(\theta)$ can be considered as an element of $\Theta(\mathcal{O}_S, \mathcal{O}_X)_p$, since $tf(\theta)(xy^2 - z^2) \equiv 0$. By this description of tf we can easily check that if we define

$$\theta_1 := \frac{1}{2}u\frac{\partial}{\partial u}, \quad \theta_2 := \frac{1}{2}v\frac{\partial}{\partial u}, \quad \theta_3 := v\frac{\partial}{\partial v}, \quad \theta_4 := uv\frac{\partial}{\partial v},$$

then $tf(\theta_i) = \omega f(\Theta_i)$ for each generator Θ_i $(1 \le i \le 4)$ of $\Theta_{S,p}$ in Lemma 3.23. This implies that for any element $\Theta \in \Theta_{S,p}$ there exists certainly an element $\theta \in \Theta_X(-logD_X)_{f^{-1}(p)}$ such that $tf(\theta) = \omega f(\Theta)$ in $\Theta(\mathcal{O}_S, \mathcal{O}_X)_p$. Further, by (3.44), if $tf(\theta) = 0$, then $\theta = 0$. Hence for any $\Theta \in \Theta_{S,p}$ an element $\theta \in \Theta_X(-\log D_X)_{f^{-1}(p)}$ with $tf(\theta) = \omega f(\Theta)$ is unique. The injectivity of the map $\Theta \longrightarrow \theta$ is proved as follows: by Lemma 3.23 an element $\Theta \in \Theta_{S,p}$ is represented as

$$\Theta := A(x,y,z)\Theta_1 + B(x,y,z)\Theta_2 + C(x,y,z)\Theta_3 + D(x,y,z)\Theta_4$$

$$(3.45) \quad = \{x\cdot A(x,y,z)+z\cdot B(x,y,z)\}(\frac{\partial}{\partial x})+\{y\cdot C(x,y,z)+z\cdot D(x,y,z)\}(\frac{\partial}{\partial y})$$
$$+\{\frac{1}{2}z\cdot A(x,y,z)+\frac{1}{2}y^2\cdot B(x,y,z)+z\cdot C(x,y,z)+xy\cdot D(x,y,z)\}(\frac{\partial}{\partial z})$$
$$(mod\ \mathcal{I}(S)_p),$$

where $A(x,y,z), B(x,y,z), C(x,y,z), D(x,y,z) \in \mathcal{O}_{\mathbb{C}^3,p}$. Hence, if we asuume $\omega f(\Theta)=0$ in $\Theta(\mathcal{O}_S,\mathcal{O}_X)_p$, then

$$u\cdot A(u^2,v,uv)+\frac{1}{2}v\cdot B(u^2,v,uv)\equiv 0, \text{ and}$$
$$v\cdot C(u^2,v,uv)+\frac{1}{2}uv\cdot D(u^2,v,uv)\equiv 0.$$

This implies that all of the coefficients of Θ in (3.45) belong to $\mathcal{I}(S)_p$, that is, $\Theta\equiv 0$ in $\Theta_{S,p}$.

(ii) Any element of $\nu_{S*}\Theta_{D_S^*}(-\Sigma\mathfrak{c}_S^*-\Sigma t_S^*)_p=\Theta_{D_S^*}(-\Sigma\mathfrak{c}_S^*-\Sigma t_S^*)_{\nu_S^{-1}(p)}$ has the form

$$\xi = a(x)x(\frac{\partial}{\partial x}).$$

Since we consider the map $g: D_X^*\longrightarrow D_S^*$ is given by $u\longrightarrow u^2=x$, for such an element ξ, we have

$$\omega g(\xi)=a(u^2)u^2\cdot g^*(\frac{\partial}{\partial x}).$$

Therefore, if we define

$$\eta := \frac{1}{2}a(u^2)u\cdot\frac{\partial}{\partial u}\in\nu_*\Theta_{D_X^*}(-\Sigma t_X^*)_p=\Theta_{D_X^*}(-\Sigma t_X^*)_{\nu^{-1}(p)},$$

$tg(\eta)=\omega g(\xi)$. Further this η is uniquely determined by ξ.

Q.E.D.

We are now going to prove Theorem 3.19.

Proof of Theorem 3.19 We will show the theorem only at a cuspidal point p of S under the same setting as in the proof of Lemma 3.24. The proofs at a double point and at a triple point are easier. It is obvious that, for an element $\Theta\in\Theta_{S,p}$ (resp. $\theta\in f_*\Theta_X(-\log D_X)_p=\Theta_X(-\log D_X)_{f^{-1}(p)}$),there exists uniquely an element $\xi\in\nu_{S*}\Theta_{D_S^*}(-\Sigma\mathfrak{c}_S^*-\Sigma t_S^*)_p=\Theta_{D_S^*}(-\Sigma\mathfrak{c}_S^*-\Sigma t_S^*)_{\nu_S^{-1}(p)}$ (resp. $\eta\in\nu_*\Theta_{D_X^*}(-\Sigma t_X^*)_p=\Theta_{D_X^*}(-\Sigma t_X^*)_{\nu^{-1}(p)}$) such that $t\nu_S(\xi)=\omega\nu_S(\Theta)$ (resp.

$t\nu_X(\eta) = \omega\nu_X(\theta)$). Hence, by Lemma 3.24, for any element $\Theta \in \Theta_{S,p}$ (resp. $(\theta, \xi) \in f_*\Theta_X(-\log D_X)_p \oplus \nu_{S*}\Theta_{D_S^*}(-\Sigma\mathfrak{c}_S^* - \Sigma t_S^*)_p$), there exists a unique element $(\theta, \xi) \in f_*\Theta_X(-\log D_X)_p \oplus \nu_{S*}\Theta_{D_S^*}(-\Sigma\mathfrak{c}_S^* - \Sigma t_S^*)_p$ with $tf(\theta) = \omega f(\Theta)$ and $t\nu_S(\xi) = \omega\nu_S(\Theta)$ (resp. $(\eta_1, \eta_2) \in \nu_*\Theta_{D_X^*}(-\Sigma t_X^*)_p^{\oplus 2}$ with $t\nu_X(\eta_1) = \omega\nu_X(\theta)$ and $tg(\eta_2) = \omega(\xi)$). The map $\widehat{\omega f} \oplus \widehat{\omega\nu_S}$ (resp. $\widehat{\omega\nu_X} - \widehat{\omega g}$) in (3.31) is defined to be the one which assignes $\Theta \in \Theta_{S,p}$ (resp. $(\theta, \xi) \in f_*\Theta_X(-\log D_X)_p \oplus \nu_{S*}\Theta_{D_S^*}(-\Sigma\mathfrak{c}_S^* - \Sigma t_S^*)_p$) to this (θ, ξ) (resp. $\eta_1 - \eta_2$).

From Lemma 3.24, (i), the injectivity of the map $\widehat{\omega f} \oplus \widehat{\omega\nu_S} : \Theta_S \to f_*\Theta_X(-logD_X) \oplus \nu_{S*}\Theta_{D_S^*}(-\Sigma\mathfrak{c}_S^* - \Sigma t_S^*)$ follows. The surjectivity at the last term in the exact sequence in (3.31) and $Im\{\widehat{\omega f} \oplus \widehat{\omega\nu_S}\} \subset Ker\{\widehat{\omega\nu_X} - \widehat{\omega g}\}$ are obvious. We will show that $Im\{\widehat{\omega f} \oplus \widehat{\omega\nu_S}\} \supset Ker\{\widehat{\omega\nu_X} - \widehat{\omega g}\}$.

Let $(\theta, \eta) \in (f_*\Theta_X(-logD_X) \oplus \nu_{S*}\Theta_{D_S^*}(-\Sigma\mathfrak{c}_S^* - \Sigma t_S^*))_p$. Then θ, η are represented as follows:

$$\theta = a(u,v)\frac{\partial}{\partial u} + v \cdot b(u,v)\frac{\partial}{\partial v}, \quad \text{and}$$
$$\eta = x \cdot c(x)\frac{\partial}{\partial x}$$

where $a(u,v), b(u,v) \in \mathcal{O}_{X,f^{-1}(p)}$ and $c(x) \in \mathcal{O}_{D_S^*,\nu^{-1}(p)}$. Remember that $\nu_X : D_X^* \to X$, $g : D_X^* \to D_S^*$ and $f : X \to S$ are given by $u \to (u,0) = (u,v)$, $u \to u^2 = x$ and $(u,v) \to (u^2, v, uv)$, respectively. Since

$$t\nu_X(a(u,0)\frac{\partial}{\partial u}) = \omega\nu_X(\theta) \quad \text{and}$$
$$tg(\frac{1}{2}uc(u^2)\frac{\partial}{\partial u}) = \omega g(\eta),$$

we have

$$\widehat{\omega\nu_X}(\theta) - \widehat{\omega g}(\eta) = \{a(u,0) - \frac{1}{2}u \cdot c(u^2)\}\frac{\partial}{\partial u}.$$

Hence if $(\theta, \eta) \in Ker(\widehat{\omega\nu_X} - \widehat{\omega g})$, then

$$a(u,0) = \frac{1}{2}u \cdot c(u^2).$$

Therefore, since $a(u,v) = a(u,0) + v \cdot a_1(u,v)$, where $a_1(u,v) \in \mathcal{O}_{X,f^{-1}(p)}$, θ is of the from

$$\theta = \{\frac{1}{2}u \cdot c(u^2) + v \cdot a_1(u,v)\}\frac{\partial}{\partial u} + v \cdot b(u,v)\frac{\partial}{\partial v}.$$

Hence

$$tf(\theta) = \{u^2 \cdot c(u^2) + 2v(u \cdot a_1(u,v))\}f^*(\frac{\partial}{\partial x}) + v \cdot b(u,v)f^*(\frac{\partial}{\partial y})$$

$$+\{\frac{1}{2}u\cdot c(u^2)+v\cdot a_1(u,v)+u\cdot b(u,v)\}vf^*(\frac{\partial}{\partial z}),$$

By Lemma 3.21 there exist $A(x,y,z), B(x,y,z), C(x,y,z)\in\mathcal{O}_{\mathbb{C}^3,o}$ such that

$$(3.46)\qquad \begin{aligned} A(u^2,v,uv)&=u^2\cdot c(u^2)+2v(u\cdot a_1(u,v))\\ B(u^2,v,uv)&=v\cdot b(u,v)\\ C(u^2,v,uv)&=\{\frac{1}{2}u\cdot c(u^2)+v\cdot a_2(u,v)+u\cdot b(u,v)\}v.\end{aligned}$$

Hence if we define

$$\Theta:=A(x,y,z)\frac{\partial}{\partial x}+B(x,y,z)\frac{\partial}{\partial y}+C(x,y,z)\frac{\partial}{\partial z},$$

then we have

$$tf(\theta)=\omega f(\Theta).$$

By (3.46) we can check that $f^*(\Theta(xy^2-z^2))\equiv 0$; that is, $\Theta\in\Theta_{S,p}$. Further

$$\omega\nu_S(\Theta)=A(x,0,0)\nu_S^*(\frac{\partial}{\partial x})=x\cdot c(x)\nu_S^*(\frac{\partial}{\partial x})=t\nu_S(\eta).$$

Hence $(\widehat{\omega f}\oplus\widehat{\omega\nu_S})(\Theta)=(\theta,\eta)$, which means that $Im(\widehat{\omega f}\oplus\widehat{\omega\nu_S})\supset Ker(\widehat{\omega\nu_X}-\widehat{\omega g})$ as required.

Q.E.D.

3.25 Corollary. $\Theta(b_\bullet)\simeq\Theta_S$.

Proof. By the definition of $\Theta(a_\bullet)$ and by Theorem 3.19,

$$\begin{aligned}&\Theta(b_\bullet)\\ &=\Theta(b_\bullet)\cap\{\Theta_S\oplus f_*\Theta_X(-logD_X)\oplus\nu_{S*}\Theta_{D_S^*}(-\Sigma\mathfrak{c}_S^*-\Sigma t_S^*)\oplus\nu_*\Theta_{D_X^*}(-\Sigma t_X^*)\}\\ &\simeq\Theta_S.\end{aligned}$$

Q.E.D.

Summarizing the results so far obtained, we have the following theorem which clarifies the relation betwen the characteristic map

$$\rho_o:T_oM\longrightarrow H^1(S,\Theta(b_\bullet))$$

of the family $\mathfrak{X}_\bullet\xrightarrow{a_\bullet}\mathfrak{S}\xrightarrow{\pi}M$ and the characteristic map

$$\sigma_o:T_oM\longrightarrow H^1(S,\Theta_S)$$

of the family $\pi:\mathfrak{S}\longrightarrow M$.

3.26 Theorem.

(i) We have the following commutative diagram:

$$
\begin{array}{ccc}
 & & H^1(S, \Theta(a_\bullet)) \\
 & \overset{\rho_o}{\nearrow} & \downarrow \\
T_oM & & \\
 & \underset{\sigma_o}{\searrow} & \\
 & & H^1(S, \Theta_S)
\end{array}
$$

(ii) If the map

$$H^0(X, \Theta_X(-logD_X)) \oplus H^0(D_S^*, \Theta_{D_S^*}(-\Sigma \mathfrak{c}_S^* - \Sigma t_S^*)) \longrightarrow H^0(D_X^*, \Theta_{D_X^*}(-\Sigma t_X^*))$$

is surjective, then we have

$$
\begin{aligned}
& H^1(S, \Theta(b_\bullet)) \simeq H^1(S, \Theta_S) \\
& \simeq Ker\{H^1(X, \Theta_X(-logD_X)) \oplus H^1(D_S^*, \Theta_{D_S^*}(-\Sigma \mathfrak{c}_S^* - \Sigma t_S^*)) \\
& \qquad\qquad \longrightarrow H^1(D_X^*, \Theta_{D_X^*}(-\Sigma t_X^*))\}.
\end{aligned}
$$

Proof. The fact that $H^1(S, \Theta(b_\bullet)) \simeq H^1(S, \Theta_S)$ follows from Corollary 3.25, and the commutativity of the diagram in (i) from the definitions of the characteristic maps ρ and σ. The assertion (ii) follows from Theorem 3.19.

Q.E.D.

3.27 Remark. The meaning of Theorem 3.26 is as follows:
$H^1(X, \Theta_X(-\log D_X))$ is the infinitesimal deformation space of a pair (X, D_X), which is isomorphic to the infinitesimal defromation space

$$\mathrm{Ext}_{\mathcal{C}}((\Omega_{D_X^*}, \Omega_X), (\mathcal{O}_{D_X^*}, \mathcal{O}_X))$$

of a holomorphic map $\nu_X : D_X^* \longrightarrow X$ (cf. [23]), since ν_X is "locally stable" (cf. [9], [22]). Note that $D_X := \nu_X(D_X^*)$ is a curve with ordinary duoble points). Deformation of a holomorphic map $\nu_X : D_X^* \longrightarrow X$ is equivalent to deformation of a cubic diagram

$$
\begin{array}{ccc}
\Sigma t_X^* & \longrightarrow & D_X^* \\
\downarrow & & \downarrow \nu_X \\
\Sigma t_X & \longrightarrow & X
\end{array}
\tag{3.47}
$$

Therefore $H^1(X, \Theta_X(-\log D_X))$ can be interpreted as the infinitesimal defromation space of the cubic diagram above. On the other hand, $H^1(D_S^*, \Theta_{D_S^*}(-\Sigma\mathfrak{c}_S^*))$ is nothing but the infinitesimal deformation space of a pair $(D_S^*, \Sigma\mathfrak{c}_S^*)$, which is isomorphic to the infinitesimal deformation space

$$\mathrm{Ext}^1_{\mathcal{C}}((\Omega_{D_X^*}, \Omega_{D_S^*}), (\mathcal{O}_{D_X^*}, \mathcal{O}_{D_S^*}))$$

of a holomorphic map $g : D_X^* \longrightarrow D_S^*$, since g is "locally stable". (Note that g is a ramified cover of degree 2). Therefore $H^1(D_S^*, \Theta_{D_S^*}(-\Sigma\mathfrak{c}_S^* - \Sigma t_S^*))$ can be interpreted as the infinitesimal deformation space of a cubic diagram

$$\begin{array}{ccc} \Sigma t_X^* & \longrightarrow & D_X^* \\ \downarrow & & \downarrow f \\ \Sigma t_S^* & \longrightarrow & D_S^* \end{array} \tag{3.48}$$

Theorem 3.26 tells that, under the assumption (ii) of the theorem, the infinitesimal deformation space $H^1(S, \Theta_S)$ (or the infinitesimal deformation space $H^1(S, \Theta(a.))$ of a cubic object $X. \longrightarrow S$) is isomorphic to the fiber product of the infinitesimal deformationspace of a cubic diagram (3.47) and that of a cubic diagram (3.48) over the infinitesimal deformation space $H^1(D_X^*, \Theta_{D_X^*}(-\Sigma t_X^*))$ of a holomorphic map $\Sigma t_X^* \longrightarrow D_X^*$ (cf. the diagram in (1.4)).

to be continued.

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