Power law in a one-dimensional random sequential packing

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Abstract

A one-dimensional random sequential packing problem is studied under the assumption that packed intervals may have arbitrarily small lengths. After generating lengths of intervals according to a probability distribution G and packing these intervals into an interval $[0, x)$, we denote by $F(x, b)$ the mean number of packed intervals whose lengths are larger than b. Then, under a general assumption on G, we obtain an explicit expression for $F(x, b)$ using G. Furthermore, when G satisfies a power law, we show that $F(x, b)$ satisfies another power law which is closely connected with that for G.

1. Introduction

The problem of one-dimensional random sequential packing has been studied by many authors such as Réyni (1958), Ney (1962), Dvoretzky and Robbins (1964), Itoh (1980), Itoh and Komaki (1992), Kimber (1994), and so on. These authors assumed that lengths of packed intervals are either equal to or larger than a positive constant. The present author deals with the problem assuming that packed intervals may have arbitrarily small lengths.

We will give an exact formulation to our problem. Consider an interval $[0, x)$, where $x \leq 1$. Let G be a probability distribution function with $G(1) = 1$ and g be its density. We begin our random packing by putting a random subinterval $[x_0, x_0 + l_0)$ into $[0, x)$, where its length l_0 is generated according to the conditional distribution $g(\cdot)/G(x)$ and its left endpoint x_0 is generated according to the uniform distribution on $[0, x - l_0)$. After the first step of packing process the remained space will be composed of two intervals $[0, x_0)$ and $[x_0 + l_0, x)$. At the second step we put two random subintervals $[x_1, x_1 + l_1), [x_2, x_2 + l_2)$, where their lengths l_1, l_2 are generated according to the conditional distributions $g(\cdot)/G(x_0), g(\cdot)/G(x - x_0 - l_0)$ respectively and there left endpoints x_1, x_2 are generated according to the uniform distributions on $[0, x_0 - l_1), [x_0 + l_0, x - l_1)$ respectively. After the second step of packing process the remained space will be composed of four intervals. At the third step we put four random subintervals each into the four intervals of the remained space, and this process of packing random subintervals will be continued indefinitely.

After infinite repetitions of the above packing process we finally arrive at the state that the original interval $[0, x)$ is completely covered by random subintervals $\{[x_i, x_i + l_i) : i = 0, 1, 2, \ldots\}$. Then we consider the random variable $N(x, b)$ that is defined as the number of subintervals $[x_i, x_i + l_i)$ such that $l_i > b$, where b is a positive constant. In this paper we investigate the expectation of $N(x, b)$, which will be denoted by $F(x, b)$. Our main interest lies in the asymptotic behaviour of $F(x, b)$ as b tends to zero.

In Section 2 we first derive an integral equation of Volterra' type that connects G and $F(x, b)$. Then, solving it in the standard manner, we obtain an explicit expression for $F(x, b)$ using G. In Section 3, under the assumption that G satisfies a power law, we investigate the asymptotic behaviour of $F(x, b)$ as b tends to zero. We show that $F(x, b)$ satisfies another power law, which is related to that for G by way of the beta function.

2. An explicit expression for $F(x,b)$

At the first step of random packing process, the following two cases occur; one is that the length of the packed subinterval is longer than or equal to b, and the other is that the length is shorter than b. Considering these cases separately, we have

$$F(x,b) = \int_b^x \frac{g(u)}{G(x)}\,du \int_0^{x-u} \{F(v,b)+F(x-u-v,b)+1\}\frac{dv}{x-u} + \int_0^b \frac{g(u)}{G(x)}\,du \int_0^{x-u} \{F(v,b)+F(x-u-v,b)\}\frac{dv}{x-u}\ . \tag{1}$$

From (1) it immediately follows that

$$F(x,b) = 1 - \frac{G(b)}{G(x)} + \frac{2}{G(x)}\int_0^x \frac{g(u)}{x-u}du \int_0^{x-u} F(v,b)dv\ .$$

Hence, noting that $F(x,b)=0$ for all $x<b$, we obtain an integral equation

$$F(x,b) = a(x,b) + \int_b^x K(x,y)F(y,b)dy\ , \tag{2}$$

where we put

$$a(x,b) = 1 - \frac{G(b)}{G(x)} \tag{3}$$

and

$$K(x,y) = \frac{2}{G(x)}\int_0^{x-y} \frac{g(u)}{x-u}du\ . \tag{4}$$

Note that the integral equation (2) is of Volterra's type. Moreover, setting $K(x,y)=0$ for $x<y$, we can see that K is continuous in the domain $D=\{(x,y): b\le y\le x\le 1\}$. Thus, the theory of integral equations assures that, if we put $K^{(1)}(x,y)=K(x,y)$ and define

$$K^{(n)}(x,y) = \int_b^x K^{(n-1)}(x,z)\,K(z,y)\,dz \tag{5}$$

for $n\ge 2$ recursively, then

$$S(x,y) = \sum_{n=1}^{\infty} K^{(n)}(x,y) \tag{6}$$

is uniformly convergent in D and the solution of the integral equation (2) is given by

$$F(x,b) = a(x,b) + \int_b^x S(x,y)\,a(y,b)\,dy\ . \tag{7}$$

Thus we obtain the following theorem.

Theorem 1. *Assume that G has a density and $G(1)=1$. Then the mean number $F(x,b)$ of packed intervals whose length are longer than b is given by* (7).

From Theorem 1 we can derive the following corollary, although it is intuitively obvious.

Corollary 1. *Under the assumption of Theorem 1,*

$$\lim_{b\to 0} F(x,b) = \infty\ .$$

Proof. Since $a(x,b)$ increases to 1 as b decreases to zero, applying the monotone convergence theorem to (7), we have

$$\lim_{b\to 0} F(x,b) = 1 + \int_0^x S(x,y)\,dy\ . \tag{8}$$

Now it can be easily seen that

$$\int_0^x K(x,y)\,dy = 2\ .$$

Then an easy induction shows that

$$\int_0^x K^{(n)}(x,y)\,dy = 2^n\ .$$

Hence

$$\int_0^x S(x,y)\,dy = \infty\ . \tag{9}$$

Therefore (8) and (9) immediately leads to the desired conclusion.

3. A power law for $F(x,b)$

In this section we investigate the asymptotic behaviour of $F(x,b)$ as b tends to zero under the assumption that the probability distribution G satisfies a power law. Before beginning the investigation we prepare several lemmas. Let us consider the equation of an unknown complex variable z,

$$B(z,1-\lambda) - \frac{z}{2(1-\lambda)} = 0\ , \tag{10}$$

where $0<\lambda<1$ and $B(\cdot,\cdot)$ denotes the beta function.

Lemma 1. The equation (10) *has only one root* $h(\lambda)$ *of order one in the right half-plane* $\{z : Re(z) > 0\}$.

Proof. In the integral which defines $B(z,1-\lambda)$, we consider the expansion

$$(1-t)^{-\lambda} = \sum_{n=0}^{\infty} c_n(\lambda)\,t^n\ , \tag{11}$$

where

$$c_n(\lambda) = (-1)^n \binom{-\lambda}{n} = \frac{\lambda(\lambda+1)(\lambda+2)\cdots(\lambda+n-1)}{n!}\ . \tag{12}$$

Then, since $c_n(\lambda)>0$, Lebesgue's dominated convergence theorem implies that

$$\begin{aligned} B(z,1-\lambda) &= \int_0^1 t^{z-1}\left\{\lim_{m\to\infty}\sum_{n=0}^{m} c_n(\lambda)\,t^n\right\} dt \\ &= \lim_{m\to\infty}\sum_{n=0}^{m}\int_0^1 t^{z-1+n}\,dt = \lim_{m\to\infty}\left\{\frac{1}{z}+\sum_{n=1}^{m}\frac{c_n(\lambda)}{z+n}\right\}\ , \end{aligned}$$

from which we obtain

$$B(z,1-\lambda) = \frac{1}{z} + \sum_{n=1}^{\infty}\frac{c_n(\lambda)}{z+n}\ . \tag{13}$$

Since, by the Stirling formula, $c_n(\lambda) = O\left(1/n^{1-\lambda}\right)$, the formula (13) gives the analytic continuation of $B(z,1-\lambda)$ in the entire plane except at $z=-n$ $(n=0,1,2,\ldots)$.

In (10), setting $z = x + \sqrt{-1}\,y$ and taking the imaginary part of (13), we have

$$\frac{-y}{x^2+y^2} + \sum_{n=1}^{\infty} c_n(\lambda) \cdot \frac{-y}{(x+n)^2+y^2} = \frac{y}{2(1-\lambda)} . \tag{14}$$

However the equation (14) can be satisfied only when $y = 0$.
Thus it suffices to consider the equation

$$B(x, 1-\lambda) = \frac{x}{2(1-\lambda)} . \tag{15}$$

However it is obvious that $B(x, 1-\lambda)$ is strictly decreasing with respect to x, $B(1, 1-\lambda) = 1/(1-\lambda)$, and $\lim_{x\to\infty} B(x, 1-\lambda) = 0$. Therefore the equation (15) has only one real root which is larger than one, which completes the proof of Lemma 1.

Now we show certain properties of the function $h(\lambda)$.

Lemma 2. *The function $h(\lambda)$ is a real-valued strictly increasing function of λ with $h(+0) = \sqrt{2}$ and $h(1-0) = 2$.*

Proof. Since, by integration by parts,

$$B(x, 1-\lambda) = \frac{x-1}{1-\lambda} B(x-1, 2-\lambda) ,$$

the equation (15) is equivalent to the equation

$$\frac{x-1}{x} B(x-1, 2-\lambda) - \frac{1}{2} = 0 , \tag{16}$$

the the left hand side of which we will denote by $w(x, \lambda)$.
Then we have

$$\begin{aligned}
\frac{\partial}{\partial x} \log w(x,\lambda) &= \frac{\partial}{\partial x} \{-\log x + \log\Gamma(x) - \log\Gamma(x+1-\lambda) + \log\Gamma(2-\lambda)\} \\
&= -\frac{1}{x} + \left\{-\gamma - \frac{1}{x} + x\sum_{n=1}^{\infty} \frac{1}{n(x+n)}\right\} \\
&\quad - \left\{-\gamma - \frac{1}{x+1-\lambda} + x\sum_{n=1}^{\infty} \frac{1}{n(x+1-\lambda+n)}\right\} \\
&= -\frac{x+2(1-\lambda)}{x(x+1-\lambda} - (1-\lambda)\sum_{n=1}^{\infty} \frac{1}{(x+n)(x+1-\lambda+n)} \\
&< 0 .
\end{aligned}$$

Thus the function $w(x, \lambda)$ is strictly decreasing with respect to x. On the other hand, it is obvious that the function $w(x, \lambda)$ is strictly increasing with respect to λ. Now suppose that $0 < \lambda_1 < \lambda_2 < 1$. Then, since $w(h(\lambda_1), \lambda_2) > w(h(\lambda_1), \lambda_1) = 0$ and $w(h(\lambda_2), \lambda_2) = 0$, we can deduce that $h(\lambda_1) < h(\lambda_2)$. Consequently we see that h is a strictly increasing function of λ.

From the monotonicity of the function $h(\lambda)$, both $h(+0)$ and $h(1-0)$ exist. Then, letting $\lambda \to 0$ in (15), we get $h(+0) = \sqrt{2}$. Furthermore, letting $\lambda \to 1-0$ in (16), we get $h(1-0) = 2$. Thus the proof of the lemma is completed.

Then we study the equation (10) in the entire plane.

Lemma 3 *In the entire plane, the equation* (10) *has roots* $\{h_n(\lambda) : n = 0, 1, 2, \ldots\}$, *each of which is of order one and* $-n-1 < h_n(\lambda) < -n$.

Proof. Even if x is negative, the argument in the proof of Lemma 1 shows again that the equation (14) can be satisfied only when $y = 0$.
Thus, taking the real part of (13) and setting $y = 0$, we have

$$B(x, 1-\lambda) = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{c_n(\lambda)}{x+n} .$$

Hence, for each n,

$$\lim_{x \to -n+0} B(x, 1-\lambda) = +\infty , \quad \lim_{x \to -n-0} B(x, 1-\lambda) = -\infty$$

and $B(x, 1-\lambda)$ is strictly decreasing in $(-n-1, -n)$. Therefore we obtain the desired conclusion.

In the proof of our main theorem below, the following function plays a key role.

$$\sigma(z) = -\frac{B(z, 1-\lambda)}{B(z, 1-\lambda) - \frac{z}{2(1-\lambda)}} . \tag{17}$$

For the function $\sigma(z)$ we need to evaluate an integral

$$I = \frac{1}{2\pi i} \int_{x_0 - i\infty}^{x_0 + i\infty} \sigma(z) t^{-z} dz , \tag{18}$$

where $0 < t < 1$ and x_0 is a sufficiently large positive constant.
To state a result of the evaluation, we use a quantity

$$m(\lambda) = \frac{h(\lambda)}{1 + 2(1-\lambda) \displaystyle\int_0^1 t^{h(\lambda)-1} (1-t)^{-\lambda} (-\log t)\, dt} . \tag{19}$$

Lemma 4 *The integral* (18) *is equal to* $m(\lambda) t^{-h(\lambda)}$.

Proof. Let R be a sufficiently large positive constant such that $R > x_0$, m be the largest positive integer such that $m < R$, and θ_0 be a positive number such that $\cos\theta_0 = x_0/R$. Furthermore, let ϵ be a sufficiently small positive constant. Now, taking account of Lemma 1 and Lemma 3, we consider a contour, displayed in the Figure 1, that is composed of the straight line L_0, parts of large semi-circles C_R^+ and C_R^-, parts of small semi-circles $C_{n,\epsilon}^+$ and $C_{n,\epsilon}^-$ for $0 \le n \le m$, and collections of line segments L_+ and L_- ;

$$\begin{aligned}
L_0 &= \{z : Re(z) = x_0, |z| \le R\} , \\
C_R^+ &= \{z = Re^{i\theta} : \theta_0 < \theta < \pi\} , C_R^- = \{z = Re^{i\theta} : -\pi < \theta < -\theta_0\} , \\
C_{n,\epsilon}^+ &= \{z = h_n(\lambda) + \epsilon e^{i\theta} : 0 < \theta < \pi\} , \\
C_{n,\epsilon}^- &= \{z = h_n(\lambda) + \epsilon e^{i\theta} : -\pi < \theta < 0\} , \\
L^+ &= \{z : -R < Re(z) < x_0, Im(z) = +0\} \cup \\
&\quad \bigcup_{n=0}^{m-1} \{z : h_{n+1}(\lambda) + \epsilon < Re(z) < h_n(\lambda) - \epsilon, Im(z) = +0\} , \\
L^- &= \{z : -R < Re(z) < x_0, Im(z) = -0\} \cup \\
&\quad \bigcup_{n=0}^{m-1} \{z : h_{n+1}(\lambda) + \epsilon < Re(z) < h_n(\lambda) - \epsilon, Im(z) = -0\} .
\end{aligned}$$

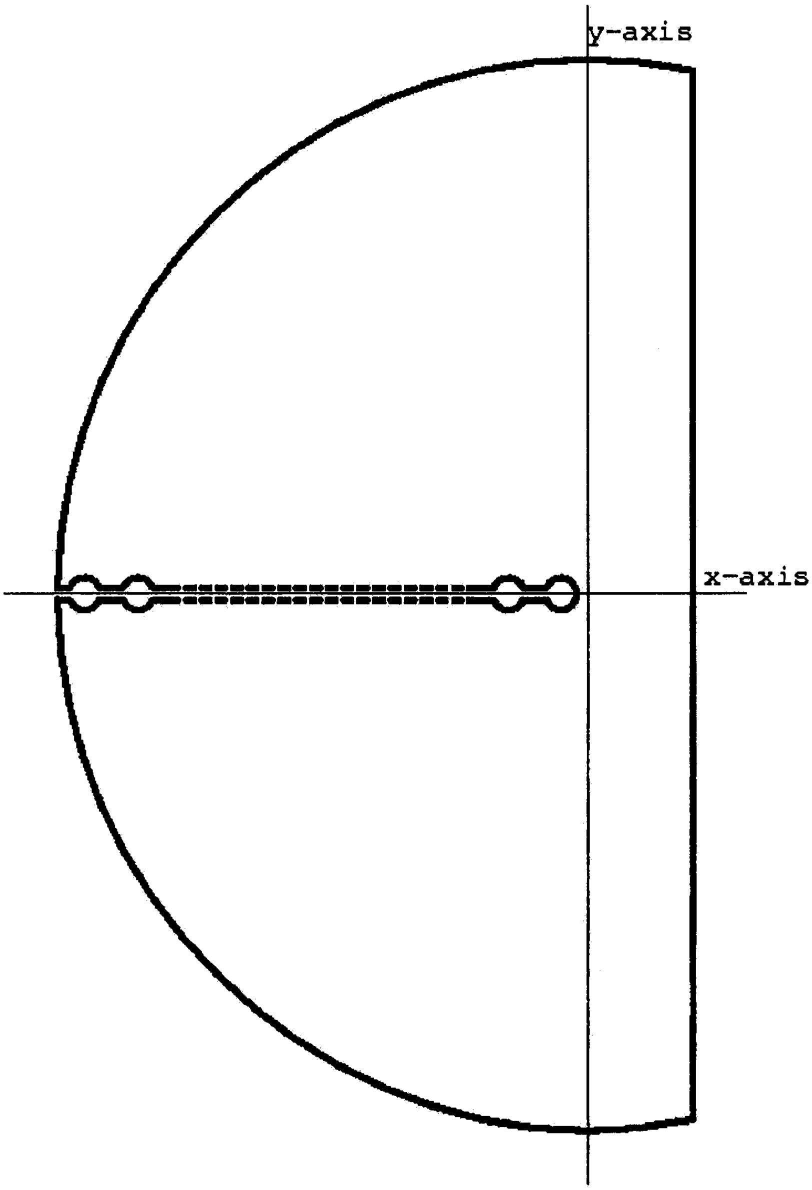

Figure 1.

Then it is obvious that

(20)
$$\frac{1}{2\pi i}\left(\int_{C^+_{n,\epsilon}}+\int_{C^-_{n,\epsilon}}\right)\sigma(z)t^{-z}dz=0$$

for each n and

(21)
$$\frac{1}{2\pi i}\left(\int_{L^+}+\int_{L^-}\right)\sigma(z)t^{-z}dz=0\ .$$

On the other hand, by the Stirling formula, as $z\to\infty$, we have $B(z,1-\lambda)\sim\Gamma(1-\lambda)/z^{1-\lambda}$, which implies that $\sigma(z)\sim 2\Gamma(2-\lambda)/z^{2-\lambda}$.
Hence

(22)
$$\frac{1}{2\pi i}\left(\int_{C^+_R}+\int_{C^-_R}\right)\sigma(z)t^{-z}dz=O\left(\frac{1}{R^{1-\lambda}}\right)\ .$$

Accordingly, combining (20), (21) and (22), we see that, when R tends to the infinity, all the contributions except by L_0 become negligible. Therefore the integral (18) can be evaluated to be the residue of $\sigma(z)t^{-z}$ at $z = h(\lambda)$. Thus

(23) $$I = -\frac{h(\lambda)}{2(1-\lambda)} \cdot t^{-h(\lambda)} \cdot \lim_{z \to h(\lambda)} \frac{z - h(\lambda)}{B(z, 1-\lambda) - \frac{z}{2(1-\lambda)}} .$$

Using L'Hospital's rule, we can see

$$\lim_{z \to h(\lambda)} \frac{B(z,1-\lambda) - \frac{z}{2(1-\lambda)}}{z - h(\lambda)} = \int_0^1 t^{h(\lambda)-1} (1-t)^{-\lambda} (\log t)\, dt - \frac{1}{2(1-\lambda)} .$$

Hence

(24) $$I = m(\lambda)\, t^{-h(\lambda)} .$$

Thus the proof of Lemma 4 is completed.

Now we state the main theorem.

Theorem 2. *Assume that*

(25) $$g(x) = (1-\lambda) \cdot \frac{1}{x^\lambda} ,$$

where $0 < \lambda < 1$. *Then*

(26) $$F(x,b) = a(x,b) + m(\lambda) \int_{\frac{b}{x}}^1 t^{-h(\lambda)}\, a(xt,b)\, dt .$$

Proof. Introduce a function

(27) $$k(t) = \int_t^1 \frac{1}{(1-u)^\lambda} \cdot \frac{du}{u} .$$

Then we can write

$$K(x,y) = 2(1-\lambda) \cdot \frac{1}{x} \cdot k\left(\frac{y}{x}\right) .$$

Hence, by induction, we have

$$K^{(n)}(x,y) = \{2(1-\lambda)\}^n \cdot \frac{1}{x} \cdot k^{(n)}\left(\frac{y}{x}\right) ,$$

where functions $k^{(n)}(t)$ are defined recursively by

(28) $$k^{(n)}(t) = \int_t^1 k^{(n-1)}(u)\, k\left(\frac{t}{u}\right) \frac{du}{u} .$$

Consequently we obtain

(29) $$S(x,y) = \frac{1}{x} s\left(\frac{y}{x}\right) ,$$

where

(30) $$s(t) = \sum_{n=1}^{\infty} \{2(1-\lambda)\}^n k^{(n)}\left(\frac{y}{x}\right) .$$

Let us consider the Mellin transforms of functions $k(t)$ and $s(t)$,

$$\kappa(z) = \int_0^1 k(t)\, t^{z-1} dt\ ,\ \sigma(z) = \int_0^1 s(t)\, t^{z-1} dt .$$

Then we can see that

$$\int_0^1 k^{(n)}(t)\, t^{z-1} dt = (\kappa(z))^n .$$

Hence, in a formal sense, we have

$$\sigma(z) = \sum_{n=1}^{\infty} \{2(1-\lambda)\}^n (\kappa(z))^n = \frac{2(1-\lambda)\kappa(z)}{1-2(1-\lambda)\kappa(z)} . \tag{31}$$

Returning to (27) we can easily get

$$\kappa(z) = \frac{1}{z} B(z, 1-\lambda) . \tag{32}$$

Substitution of (32) into (31) gives the expression (17). Moreover, recalling the asymptotic behaviour of $B(z, 1-\lambda)$, we can see that there is a positive constant x_0 such that the series (31) converges uniformly for $Re(z) \geq x_0$.

Now we apply Mellin's inversion formula to $\sigma(z)$,

$$s(t) = \frac{1}{2\pi i} \int_{x_0 - i\infty}^{x_0 + i\infty} \sigma(z) t^{-z} dz .$$

Then Lemma 4 shows that

$$s(t) = m(\lambda)\, t^{-h(\lambda)} . \tag{33}$$

Finally Theorem 1, combined with (29), implies that

$$\begin{aligned} F(x,b) &= a(x,b) + \frac{1}{x} \int_b^x s\left(\frac{y}{x}\right) a(y,b)\, dy \\ &= a(x,b) + \int_{\frac{b}{x}}^1 s(t)\, a(xt,b)\, dt . \end{aligned}$$

Therefore, using (33), we obtain (26). Thus the proof of Theorem 2 is completed.

Using Theorem 2, as the asymptotic behaviour of $F(x,b)$, we can obtain a power law.

Corollary 2. *As b tends to zero,*

$$F(x,b) = m(\lambda) \left(\frac{1}{h(\lambda)-1} + \frac{1}{h(\lambda)-\lambda} \right) \left(\frac{x}{b} \right)^{h(\lambda)-1} + O(1) .$$

Proof. From (26) it follows that

$$\begin{aligned} F(x,b) &= 1 - \left(\frac{b}{x}\right)^{1-\lambda} + m(\lambda) \int_{\frac{b}{x}}^1 t^{-h(\lambda)}\, dt - m(\lambda) \cdot \left(\frac{b}{x}\right)^{1-\lambda} \int_{\frac{b}{x}}^1 t^{-(h(\lambda)+1-\lambda)}\, dt \\ &= 1 - \left(\frac{b}{x}\right)^{1-\lambda} + \frac{m(\lambda)}{h(\lambda)-1} \left\{ \left(\frac{x}{b}\right)^{h(\lambda)-1} - 1 \right\} \\ &\quad + \frac{m(\lambda)}{h(\lambda)-\lambda} \left\{ \left(\frac{x}{b}\right)^{h(\lambda)-1} - \left(\frac{b}{x}\right)^{1-\lambda} \right\} . \end{aligned}$$

Hence we obtain the desired conclusion.

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