Regular Rod Packing in Four-Dimensional Space.

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Abstract

Regular periodic packing of rectangular rods with infinite height is studied in the four-dimensional Euclidean space. We assume that directions of axes of rods are parallel to one of the coordinates axes of the space, and 'coordinates' that can specifiy positions of rods are always integers. Then we construct two configurations of rods; one is a configuration of 'slim' rods for which the packing density of rods is unity; another is a configuration of 'fat' rods for which the packing density of rods is 3/4. The full packing density for 'slim' rods is a new phenomenon that never occur in the usual three-dimensional space, while the packing density 3/4 for 'fat' rods coincides with that for rod packing in the three-dimensional space.

1 Introduction

In this short paper we present several examples of regular configuration generated by rod packing in the four-dimensional Euclidean space. Readers who are interested in research on rod packing general, may review, for example, O'Keeffe and Andersson (1977), Teshima et al. (2000), O'Keeffe et al. (2001), Bezdek and Kuperberg (1990), Isokawa (2001) and Isokawa (2006).

We use the word 'regular' to mean 'periodic' as well as 'non-random', 'rod' to mean a rectangular prism with large (may be infinite) height. Consider a cube $[0,a]^4$, where a is an integer. We treat two types of rods; one is a Cartesian product of a two-dimensional base and a two-dimensional height, where base is a unit square parallel to the two-dimensional Cartesian plane, and height coincides with $[0,a]^2$; another is a Cartesian product of a three-dimensional base and an one-dimensional height, which is a unit cube parallel to the three-dimensional Cartesian hyperplane, and height coinsides with $[0,a]$. Let us call the former type rod a 'fat' rod, and the latter a 'slim' rod.

Suppose that the Euclidean space be equipped with the four Cartesian axes i-axis (i = 0,1,2,3). Then every fat rod has one of the following forms:

(f1) $[x_0, x_0+1] \times [x_1, x_1+1] \times [0,a] \times [0,a]$,
(f2) $[x_0, x_0+1] \times [0,a] \times [x_2, x_2+1] \times [0,a]$,
(f3) $[x_0, x_0+1] \times [0,a] \times [0,a] \times [x_3, x_3+1]$,
(f4) $[0,a] \times [x_1, x_1+1] \times [x_2, x_2+1] \times [0,a]$,
(f5) $[0,a] \times [x_1, x_1+1] \times [0,a] \times [x_3, x_3+1]$,
(f6) $[0,a] \times [0,a] \times [x_2, x_2+1] \times [x_3, x_3+1]$.

A rod of the form (f1) will be called $(0,1)$-type with coordinates (x_0, x_1). Rods of other types ($(0,2)$-type, $(0,3)$-type, $(1,2)$-type, $(1,3)$-type, $(2,3)$-type) and their coordinates are similarly defined. *Fat rods are one of the six types.*

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On the other hand, every slim rod has one of the following forms:

(s1) $[x_0, x_0+1] \times [x_1, x_1+1] \times [x_2, x_2+1] \times [0, a]$,
(s2) $[x_0, x_0+1] \times [x_1, x_1+1] \times [0, a] \times [x_3, x_3+1]$,
(s3) $[x_0, x_0+1] \times [0, a] \times [x_2, x_2+1] \times [x_3, x_3+1]$,
(s4) $[0, a] \times [x_1, x_1+1] \times [x_2, x_2+1] \times [x_3, x_3+1]$.

A rod the form (s1) will be called $(0,1,2)$-type with coordinates (x_0, x_1, x_2). Rods of other types ($(0,1,3)$-type, $(0,2,3)$-type, $(1,2,3)$-type) and their coordinates are similarly defined. *Slim rods are one of the four types.*

We assume that components x_0, x_1, x_2, x_3 of coordinates are integers. Obviously they belong to the set $A = \{0, 1, 2, \cdots, a-1\}$.

We consider a 'final' configuration of rods where no more rod can be packed. In ths paper we show the following facts:

(1) fat rod packing can generate two non-trivial configurations; one has the full packing density ie $[0, a]^4$ is completely occupied by rods; another is periodic and has the packing density $3/4$.

(2) slim rod packing can generate a non-trivial configuration which is periodic, is isotropic, and has the full packing density. Here 'non-trivial' is of course the negation of 'trivial', which means all axes packed rods have the same direction. Moreover 'isotropic' means that the numbers of rods of specified types are the same for all types.

2 Fat rod packing

If a $(0,1)$-rod with coordinates (x_0, x_1) is packed, they it is impossible to pack any $(2,3)$-rod, because the set $[x_0, x_0+1] \times [x_1, x_1+1] \times [0, a] \times [0, a]$ intersects the set $[0, a] \times [0, a] \times [x_2, x_2+1] \times [x_3, x_3+1]$ for any (x_2, x_3). Similarly concurrent packing of $(0,2)$-rod and $(1,3)$-rod is impossible, and so is concurrent packing of $(0,3)$-rod and $(1,2)$-rod.

Consequently, without loss of generality, it is sufficient to consider the following two cases:

(a) packing only $(0,1)$-rods, $(0,2)$-rods, and $(0,3)$-rods,
(b) packing only $(0,1)$-rods, $(0,2)$-rods, and $(1,2)$-rods.

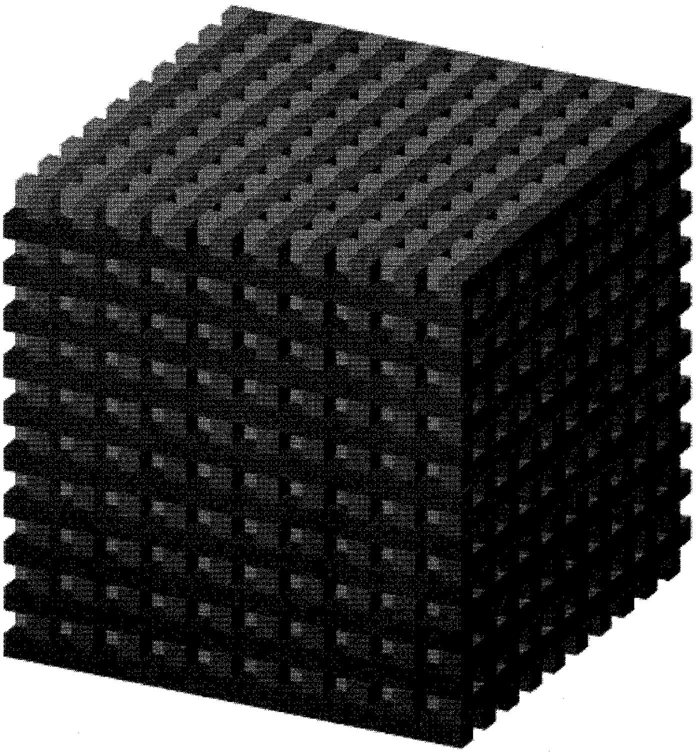
In the case (a), consider the following sets,

$$\begin{aligned} I_1 &= \{x_0 : (0,1) - \text{rod with coordinates}(x_0, x_1)\text{is packed}\}, \\ I_2 &= \{x_0 : (0,2) - \text{rod with coordinates}(x_0, x_2)\text{is packed}\}, \\ I_3 &= \{x_0 : (0,3) - \text{rod with coordinates}(x_0, x_3)\text{is packed}\}, \end{aligned}$$

Then it is necessary that these sets are disjoint each other, otherwise two rods intersect. Furthermore, since we consider a 'final' configuration of rods where no more rod can be packed, it is also necessary that $I \cup J \cup K = A$. Note that the number of $(0,1)$-rods is equal to $a|I|$, the number of $(0,2)$-rods $a|J|$, and the number of $(0,3)$-rods $a|K|$. Hence we obtain

$$\text{packing density} = \frac{(a|I| + a|J| + a|K|)a^2}{a^4} = 1.$$

In the case (b), consider sections of rods with the hyperplane $x_3 = 0$. Then packing of fat rods in the four-dimensional space reduces to packing in the three-dimensional space. In the three-dimensional space several configurations are well-known, of which the most simple is the following periodic configuration with packing density $3/4$.



3 Slim rod packing

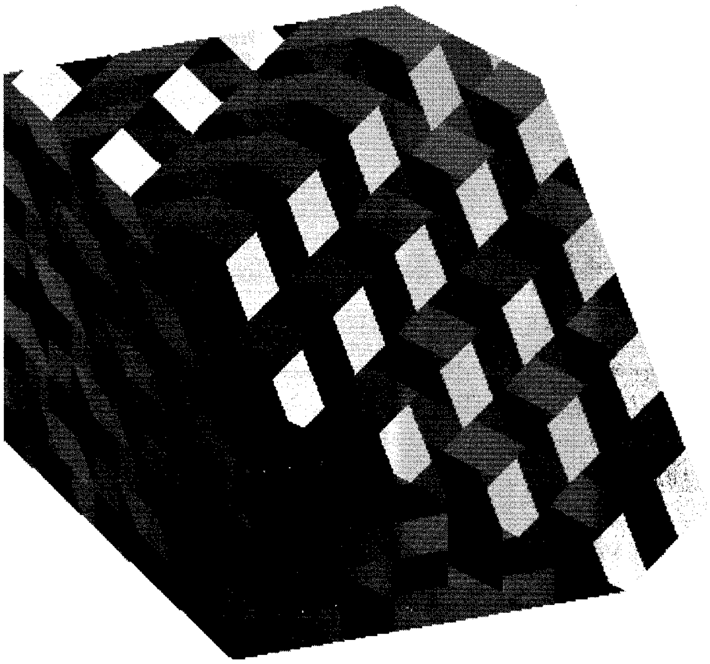
In this paper we try to find a periodic configuration with period 2. We have checked all configurations for $a = 2$ case 'one by one'. Then we arrive at the conclusion that there are essentially only two configurations as follows:

	$(0,1,2)$-type	$(0,1,3)$-type	$(0,2,3)$-type	$(1,2,3)$-type
(a)	$(0,0,0),(1,1,1)$	$(0,1,0),(1,0,1)$	$(0,1,1),(1,0,0)$	$(0,1,0),(1,0,1)$
(b)	$(0,0,0),(1,1,1)$	$(0,1,0),(0,1,1)$	$(1,0,0),(1,0,1)$	$(0,1,0),(0,1,1)$

Here 'essentially' means that we identify two configurations if one can be transformed to another by some permutation of the coordinates axes.

The above two configurations (a) and (b) can be extended to periodic configurations for any a. It is aesy to confirm that they have the full packing density.

The following figure shows a 3-dimensional section of configuration of regular packing of 'slim' rods by a hyperplane $x_0 + x_1 + x_2 + x_3 = a$.



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